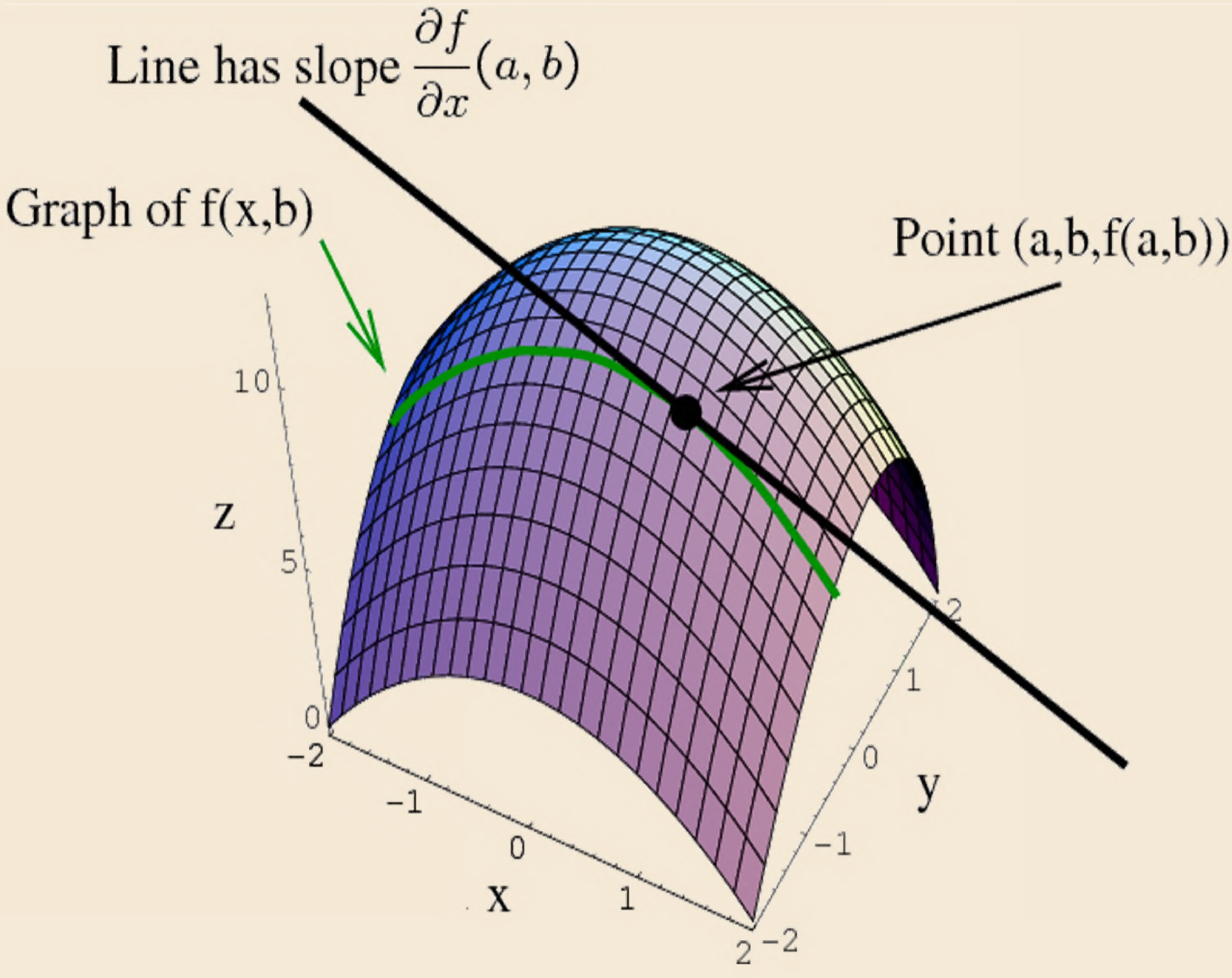
Continuity & Differentiability



Solved Exercices

Content

TOPIC	PAGE NO.
◆ THEORY WITH SOLVED EXAMPLES	3 – 28
◆ EXERCISE - I	29 – 38
◆ EXERCISE - II	39 – 42
◆ EXERCISE - III	43 – 47
◆ EXERCISE - IV	48 – 51
◆ EXERCISE - V	52 – 54
◆ ANSWER KEY	55 – 56

JEE Syllabus :

continuity of a function, continuity of the sum, difference, product and quotient of two functions, continuity of composite functions, intermediate value property of continuous functions

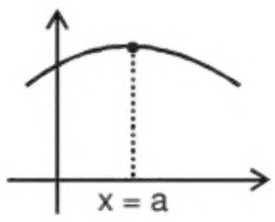
A. DEFINITION OF CONTINUITY

Continuity at a Point : – A function f is **continuous at** c if the following three conditions are met.

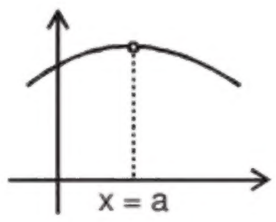
(i) f(x) is defined. **(ii)** $\lim\limits_{x \to c} f(x)$ exists. **(iii)** $\lim\limits_{x \to c} f(x) = f(c)$.

In other words function f(x) is said to be continuous at $x = c$, if $\underset{x \to c}{\text{Limit}}\ f(x) = f(c)$.

Symbolically f is continuous at $x = c$ if $\underset{h \to 0}{\text{Limit}}\ f(c - h) = \underset{h \to 0}{\text{Limit}}\ f(c+h) = f(c)$.



(continuous)



(Discontinuous)

One-sided Continuity :

A function f defined in some neighbourhood of a point c for $c \le c$ is said to be continuous at c from the left if $\lim\limits_{x \to c^-} f(x) = f(c)$.

A function f defined in some neigbourhood of a point c for $x \ge c$ is said to be continuous at c from the right if $\lim\limits_{x \to c^+} f(x) = f(c)$.

One-sided continuity is a collective term for functions continuous from the left or from the right.

If the function f is continuous at c, then it is continuous at c from the left and from the right . Conversely, if the function f is continuous at c from the left and from the right, then $\lim\limits_{x \to c} f(x)$ exists & $\lim\limits_{x \to c} f(x) = f(c)$.

The last equality means that f is continuous at c.

If one of the one-sided limits does not exist, then $\lim\limits_{x \to c} f(x)$ does not exist either. In this case, the point c is a discontinuity in the function, since the continuity condition is not met.

Continuity In An Interval :

(a) A function f is said to be continuous in an open interval (a , b) if f is continuous at each & every point $\in$ (a , b).

(b) A function f is said to be continuous in a closed interval $[a, b]$ if :

(i) f is continuous in the open interval (a , b) &

(ii) f is right continuous at **'a'** i.e. $\underset{x \to a^+}{\text{Limit}}\ f(x) = f(a)$ = a finite quantity.

(iii) f is left continuous at **'b'** i.e. $\underset{x \to b^-}{\text{Limit}}\ f(x) = f(b)$ = a finite quantity.

A function f can be discontinuous due to any of the following three reasons :

(i) $\underset{x\to c}{\text{Limit}}$ f(x) does not exist i.e. $\underset{x\to c^-}{\text{Limit}} f(x) \neq \underset{x\to c^+}{\text{Limit}} f(x)$

(ii) f(x) is not defined at x= c **(iii)** $\underset{x\to c}{\text{Limit}} f(x) \neq f(c)$

Geometrically, the graph of the function will exhibit a break at x= c.

Ex.1 Test the following functions for continuity

(a) $\dfrac{2x^5 - 8x^2 + 11}{x^4 + 4x^3 + 8x^2 + 8x + 4}$ **(b)** $f(x) = \dfrac{3\sin^3 x + \cos^2 x + 1}{4\cos x - 2}$

Sol. **(a)** A function representing a ratio of two continous functions will be (polynomials in this case) discontinuous only at points for which the denominator zero. But in this case

$(x^4 + 4x^3 + 8x^2 + 8x + 4) = (x^2 + 2x + 2)^2 = [(x + 1)^2 + 1]^2 > 0$ (always greater than zero)

Hence f(x) is continuous throughout the entire real line.

(b) The function f(x) suffers discontinuities only at points for which the denominator is equal to zero i.e. $4\cos x - 2 = 0$ or $\cos x = 1/2 \Rightarrow x = x_n = \pm \pi/3 + 2n\pi$ (n = 0, ±1, ±2...)

Thus the function f(x) is continuous everywhere, except at the point x_n.

Ex.2 The function $f(x) = \dfrac{2 - (256 - 7x)^{1/8}}{(5x + 32)^{1/5} - 2}$, $x \neq 0$ is continuous everywhere then find the value of f(0).

Sol. Since f(x) is continuous

$\therefore$ $f(0) = \text{R.H.L.} = \lim\limits_{x\to 0+} f(x) = \lim\limits_{h\to 0} f(0 + h)$

$$= -\lim_{h\to 0}\frac{(256-7h)^{1/8}-(256)^{1/8}}{(5h+32)^{1/5}-(32)^{1/5}} = -\lim_{h\to 0}\frac{\dfrac{(256-7h)^{1/8}-(256)^{1/8}}{(256-7h)-256}.(-7h)}{\dfrac{(5h+32)^{1/5}-(32)^{1/5}}{(5h+32)-32}.(5h)} = \frac{7}{5}\lim_{h\to 0}\frac{\dfrac{(256-7h)^{1/8}-(256)^{1/8}}{(256-7h)-256}}{\dfrac{(5h+32)^{1/5}-(32)^{1/5}}{(5h+32)-32}}$$

$$= \frac{7}{5}.\frac{\frac{1}{8}.(256)^{1/8-1}}{\frac{1}{5}.(32)^{1/5-1}} = \frac{7}{8}.\frac{(2)^{-7}}{(2)^{-4}} = \frac{7}{8}.\frac{1}{2^3} = \frac{7}{64} \Rightarrow f(0) = \frac{7}{64}. \quad \left\{\because \lim_{x\to a}\frac{x^n - a^n}{x-a} - na^{n-1}\right\}$$

Ex.3 Let $f(x) = \begin{cases} -2\sin x & \text{if} \quad x \leq -\pi/2 \\ A\sin x + B & \text{if} \quad -\dfrac{\pi}{2} < x < \pi/2 \\ \cos x & \text{if} \quad x \geq \pi/2 \end{cases}$ Find A and B so as to make the function continuous.

Sol. At $x = -\pi/2$

L.H.L. $= \lim\limits_{x\to -\frac{\pi^-}{2}} (-2\sin x)$ R.H.L. $= \lim\limits_{x\to -\frac{\pi^+}{2}} A\sin x + B$

Replace x by $-\frac{\pi}{2}$ – h where $h \to 0$ Replace x by $-\frac{\pi}{2}$ + h where $h \to 0$.

$\lim\limits_{h\to 0} -2\sin\left(-\frac{\pi}{2}-h\right) = 2$ $= \lim\limits_{h\to 0} A\sin\left(-\frac{\pi}{2}+h\right) + B = B - A$

So B – A = 2 ...(i)

At x = π/2

L.H.L. = $\lim\limits_{x\to \frac{\pi^-}{2}} A\sin x + B$ R.H.L. = $\lim\limits_{x\to \frac{\pi^+}{2}} \cos x$

Replace x by $\frac{\pi}{2}$ – h Replace x by $\frac{\pi}{2}$ + h

where $h \to 0$ where $h \to 0$

$= \lim\limits_{h\to 0} A\sin\left(\frac{\pi}{2}-h\right) + B = A + B$ $= \lim\limits_{h\to 0} \cos\left(\frac{\pi}{2}+h\right) = 0$

So A + B = 0 ...(ii)

Solving (i) & (ii), B = 1, A = –1

Ex.4 Test the continuity of f(x) at x = 0 if $f(x) = \begin{cases} (x+1)^{2-\left(\frac{1}{|x|}+\frac{1}{x}\right)} & , \ x \neq 0 \\ 0 & , \ x = 0 \end{cases}$

Sol. For x < 0,

L.H.L. = $\lim\limits_{x\to 0-} f(x) = \lim\limits_{h\to 0} f(0-h) = \lim\limits_{h\to 0} (0-h+1)^{2-\left(\frac{1}{|0-h|}+\frac{1}{(0-h)}\right)} = \lim\limits_{h\to 0} (1-h)^2 = (1-0)^2 = 1$

f(0) = 0. & R.H.L = $\lim\limits_{x\to 0+} f(x) = \lim\limits_{h\to 0} f(0+h) = \lim\limits_{h\to 0} (h+1)^{2-\left(\frac{1}{|h|}+\frac{1}{h}\right)} = \lim\limits_{h\to 0} (h+1)^{2-\frac{2}{h}} = 1^{-\infty} = 1$

∴ L.H.L. = R.H.L. ≠ f(0) Hence f(x) is discontinuous at x = 0.

Ex.5 If f(x) be continuous function for all real values of x and satisfies;

$x^2 + \{f(x) - 2\}x + 2\sqrt{3} - 3 - \sqrt{3}\,.\,f(x) = 0,\ \forall\ x \in R$. Then find the value of $f(\sqrt{3})$.

Sol. As f(x) is continuous for all x ∈ R.

Thus, $\lim\limits_{x\to\sqrt{3}} f(x) = f(\sqrt{3})$ where $f(x) = \dfrac{x^2-2x+2\sqrt{3}-3}{\sqrt{3}-x},\ x \neq \sqrt{3}$

$\lim\limits_{x\to\sqrt{3}} f(x) = \lim\limits_{x\to\sqrt{3}} \dfrac{x^2-2x+2\sqrt{3}-3}{\sqrt{3}-x}$ $= \lim\limits_{x\to\sqrt{3}} \dfrac{(2-\sqrt{3}-x)(\sqrt{3}-x)}{(\sqrt{3}-x)} = 2(I-\sqrt{3})$

$f(\sqrt{3}) = 2(1-\sqrt{3})$.

Ex.6 Let $f(x) = \begin{cases} (1+|\sin x|^{a/(|\sin x|)} & , -\frac{\pi}{6} < x < 0 \\ b & , x = 0 \\ e^{\tan 2x/\tan 3x} & , x > 0 \end{cases}$ Determine a and b such that f is continuous at x = 0.

Sol. For x < 0,

L.H.L. = $\lim_{x \to 0-} f(x) = \lim_{h \to 0} f(0 - h) = \lim_{h \to 0} (1 + |\sin(0 - h)|)^{a/(|\sin(0-h)|)}$

$= \lim_{h \to 0} (1 + \sin h)^{a/(\sin h)} = e^{\lim_{h \to 0} (1+\sinh-1).a/(\sinh)} = e^a$

For x = 0, f(0) = b

For x > 0, R.H.L. = $\lim_{x \to 0+} f(x) = \lim_{h \to 0} f(0 + h) = e^{\lim_{h \to 0} \frac{\tan 2h}{\tan 3h}} = e^{\lim_{h \to 0} \frac{\tan 2h/2h}{\tan 3h/3h} \cdot \frac{2}{3}} = e^{(1/1) \cdot (2/3)} = e^{2/3}$

$\because$ f(x) is continuous at x = 0 $\therefore$ L.H.L. = R.H.L. = f(0)

$\Rightarrow$ $e^a = e^{2/3} = b$ Hence a = 2/3 and b = $e^{2/3}$.

Ex.7 If $f(x) = \frac{A\cos x + Bx\sin x - 5}{x^4}$ $(x \neq 0)$ is continuous at x = 0, then find the value of A and B . Also find f (0).

Sol. For continuity $\underset{x \to 0}{\text{Limit}}\, f(x) = f(0)$

Now $\underset{x \to 0}{\text{Limit}} \frac{A\cos x + Bx\sin x - 5}{x^4}$ as $x \to 0$; Numerator $\to A - 5$ and Denominator $\to 0$.

Hence $A - 5 = 0 \Rightarrow A = 5$

Hence $\underset{x \to 0}{\text{Limit}} \frac{Bx\sin x - 5(1-\cos x)}{x^4} = \underset{x \to 0}{\text{Limit}} \frac{B.\frac{\sin x}{x} - \frac{5}{1+\cos x}\frac{\sin^2 x}{x^2}}{x^2}$

as $x \to 0$; Numerator $\to B - \frac{5}{2}$ and Denominator $\to 0 \Rightarrow B = \frac{5}{2}$

Hence $l = \frac{5}{2} \underset{x \to 0}{\text{Limit}} \frac{x\sin x - 2(1-\cos x)}{x^4} = \frac{5}{2} \underset{x \to 0}{\text{Limit}} \frac{2x\sin\frac{x}{2}\cos\frac{x}{2} - 4\sin^2\frac{x}{2}}{x^4}$

$= \frac{5}{2} \underset{x \to 0}{\text{Limit}} \frac{2\sin\frac{x}{2}}{x} \underset{x \to 0}{\text{Limit}} \frac{x\cos\frac{x}{2} - 2\cos\frac{x}{2}}{x^3}$ Let $x = 2\theta$

$= \frac{5}{16} \underset{\theta \to 0}{\text{Limit}} \frac{2\theta\cos\theta - 2\sin\theta}{\theta^3} = \frac{5}{16} \underset{\theta \to 0}{\text{Limit}}\, 2\cos\theta \frac{(\theta - \tan\theta)}{\theta^3} = \frac{5}{8} \underset{\theta \to 0}{\text{Limit}} \frac{\theta - \tan\theta}{\theta^3} = \frac{5}{8}\left(-\frac{1}{3}\right) = -\frac{5}{24}$

Ex.8 Discuss the continuity of the function $f(x) = \begin{cases} \dfrac{a^{2[x]+\{x\}}-1}{2[x]+\{x\}} & , \quad x \neq 0 \\ \log_e a & , \quad x = 0 \end{cases}$ $\quad (a \neq 1)$

at x = 0, where [x] and {x} are the greatest integer function and fractional part of x respectively.

Sol. Value of function = $f(0) = \log_e a$

$$\text{L.H.L.} = \lim_{x \to 0-} f(x) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} \frac{a^{2[0-h]+\{0-h\}}-1}{2[0-h]+\{0-h\}} = \lim_{h \to 0} \frac{a^{2[0-h]+\{-1+(1-h)\}}-1}{2[0-h]+\{-1+(1-h)\}} = \frac{a^{-1}-1}{-1} = 1-\frac{1}{a}$$

$$\text{and R.H.L.} = \lim_{x \to 0+} f(x) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} \frac{a^{2[0+h]+\{0+h\}-1}}{2[0+h]+\{0+h\}} = \lim_{h \to 0} \frac{a^{0+h}-1}{0+h} = \lim_{h \to 0} \frac{a^h-1}{h} = \log_e a$$

$\therefore$ L.H.L. $\neq$ R.H.L. = f(0) Hence f(x) is discontinuous at x = 0.

Ex.9 Let $f(x) = \begin{cases} \dfrac{1+a\cos 2x+b\cos 4x}{x^2 \sin^2 x} & \text{if } x \neq 0 \\ c & \text{if } x = 0 \end{cases}$ If $f(x)$ is continuous at x = 0, then find the value of $(b + c)^3$ – 3a.

Sol. $\lim\limits_{x \to 0} \dfrac{1+a\cos 2x+b\cos 4x}{x^4}$ as $x \to 0$, $N^r \to 1 + a + b$ $\quad D^r \to 0$

for existence of limit a + b + 1 = 0 $\quad \therefore \quad c = \lim\limits_{x \to 0} \dfrac{a\cos 2x+b\cos 4x-(a+b)}{x^4}$ .....(2)

$$= -\lim_{x \to 0} \frac{\dfrac{a(1-\cos 2x)}{x^2}+\dfrac{b(1-\cos 4x)}{x^2}}{x^2}$$

limit of N^r $\Rightarrow$ 2a + 8b = 0 $\Rightarrow$ a = – 4b

hence – 4b + b = – 1 $\Rightarrow$ $b = \dfrac{1}{3}$ and $a = -\dfrac{4}{3}$

hence $c = \lim\limits_{x \to 0} \dfrac{4(1-\cos 2x)-(1-\cos 4x)}{3x^2} = \dfrac{8\sin^2 x - 2\sin^2 2x}{3x^4} = \dfrac{8\sin^2 x - 8\sin^2 x\cos^2 x}{3x^4}$

$\dfrac{8}{} \cdot \dfrac{2}{2} \cdot \dfrac{2}{2} \quad \dfrac{8}{}$

Ex.10 Let $f(x) = \begin{cases} \dfrac{a(1-x\sin x)+b\cos x+5}{x^2} & x<0 \\ 3 & x=0 \\ \left(1+\left(\dfrac{cx+dx^3}{x^2}\right)\right)^{\frac{1}{x}} & x>0 \end{cases}$. If f is continuous at $x = 0$, then find the values of a, b, c & d .

Sol. $f(0^-) = \underset{x \to 0}{\text{Limit}} \dfrac{a(1-x\ \sin x) + b\cos x + 5}{x^2}$ for existence of limit $a + b + 5 = 0$

$= \underset{x \to 0}{\text{Limit}} \dfrac{a(1-x\ \sin x) - (a+5)\cos x + 5}{x^2} \quad = \underset{x \to 0}{\text{Limit}} \dfrac{a(1-\cos x) + 5(1-\cos x) - ax\sin x}{x^2}$

$= \dfrac{a}{2} + \dfrac{5}{2} - a = 3 \Rightarrow a = -1 \quad \Rightarrow \quad b = -4$

$f(0^+) = \underset{x \to 0}{\text{Limit}} \left[1 + \dfrac{x(c+dx^2)}{x^2}\right]^{1/x}$ for existence of limit $c = 0$

$\underset{x \to 0}{\text{Limit}} (1 + dx)^{1/x} = e^{\underset{x \to 0}{\text{Limit}} \frac{1}{x} dx} = e^d = 3 \quad \Rightarrow \quad d = \ell n\, 3$

Ex.11 The function, $f(x) = \dfrac{e^{2x}-1-x(e^{2x}+1)}{x^3}$ is not defined at $x = 0$. What should be the value of $f(x)$ so that $f(x)$ is continuous at $x = 0$.

Sol. $l = \underset{x\to 0}{\text{Limit}} \dfrac{e^{2x}-1-x(e^{2x}+1)}{x^3}$ Put $x = 3t$

$= \underset{t\to 0}{\text{Limit}} \dfrac{e^{6t}-1-3t(e^{6t}+1)}{27t^3} \quad = \underset{t\to 0}{\text{Limit}} \dfrac{(e^{2t}-1)^3+3e^{2t}(e^{2t}-1)-3t(e^{6t}+1)}{27t^3}$

$= \underset{t\to 0}{\text{Limit}} \dfrac{(e^{2t}-1)^3+3e^{2t}\left[e^{2t}-1-t(e^{2t}+1)\right]+3t\left[e^{2t}(e^{2t}+1)-e^{6t}-1\right]}{27t^3}$

$= \underset{t\to 0}{\text{Limit}} \dfrac{(e^{2t}-1)^3}{27t^3} + \dfrac{1}{9} \underset{t\to 0}{\text{Limit}}\ e^{2t} \times l - \underset{t\to 0}{\text{Limit}} \dfrac{1}{9t^2} (e^{2t}-1)(e^{4t}-1)$

$\Rightarrow \dfrac{8\ell}{9} = \dfrac{8}{27} - \dfrac{8}{9} \underset{t\to 0}{\text{Limit}} \left(\dfrac{e^{2t}-1}{2t}\right) \times \left(\dfrac{e^{4t}-1}{4t}\right) \Rightarrow l = \dfrac{1}{3} - 1 = -\dfrac{2}{3}$

Ex.12 Let $f(x) = x^3 - 3x^2 + 6 \ \forall\ x \in R$ and $g(x) = \begin{cases} \max\{f(t) : x+1 \le t \le x+2, -3 \le x < 0\} \\ 1-x, \quad \text{for} \quad x \ge 0 \end{cases}$

Test continuity of g (x) for $x \in [-3, 1]$.

Sol. Since $f(x) = x^3 - 3x^2 + 6 \quad \Rightarrow \quad f'(x) = 3x^2 - 6x \quad = 3x(x-2)$

for maxima and minima $f'(x) = 0$

$\therefore \quad x = 0, 2$

$f''(x) = 6x - 6$

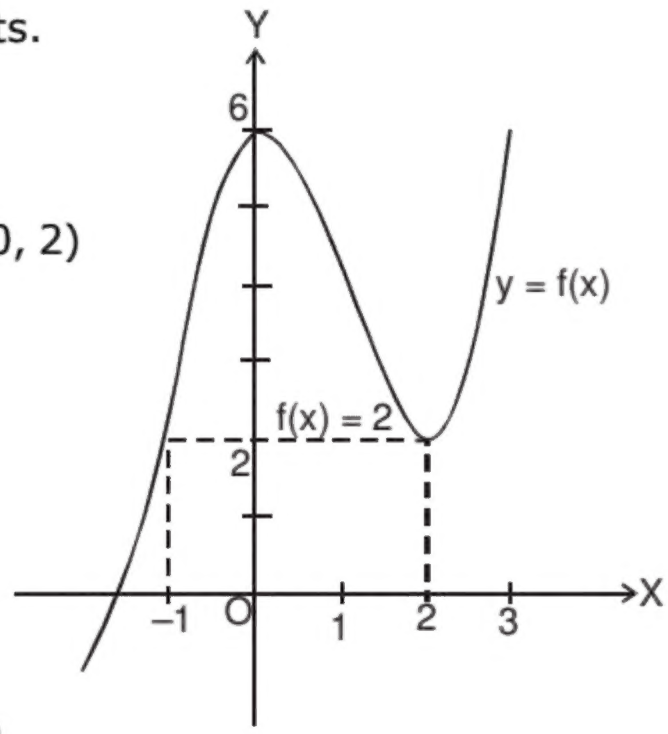
$f''(0) = -6 < 0$ (local maxima at $x = 0$)

$f''(2) = 6 > 0$ (local minima at $x = 2$)

$x^3 - 3x^2 + 6 = 0$ has maximum 2 positive and 1 negative real roots.

$f(0) = 6$.

Now graph of f(x) is :



Clearly f(x) is increasing in $(-\infty, 0) \cup (2, \infty)$ and decreasing in (0, 2)

$\Rightarrow \quad x + 2 < 0 \Rightarrow x < -2 \quad \Rightarrow \quad -3 \le x < -2$

$\Rightarrow \quad -2 \le x + 1 < -1$ and $-1 \le x + 2 < 0$

in both cases f(x) increases (maximum) of $g(x) = f(x + 2)$

$\therefore \quad g(x) = f(x + 2);\ -3 \le x < -2$ ...(1)

and if $x + 1 < 0$ and $0 \le x + 2 < 2$

$\Rightarrow \quad -2 \le x < -1$ then $g(x) = f(0)$

Now for $x + 1 \ge 0$ and $x + 2 < 2 \Rightarrow \quad -1 \le x < 0,\ g(x) = f(x + 1)$

Hence $g(x) = \begin{cases} f(x+2) & ; \ -3 \le x < -2 \\ f(0) & ; \ -2 \le x < -1 \\ f(x+1) & ; \ -1 \le x < +0 \\ 1-x & ; \quad x \ge 0 \end{cases}$ Hence g(x) is continuous in the interval [–3, 1].

Ex.13 Let y = f(x) be defined parametrically as $y = t^2 + t\,|t|$, $x = 2t - |t|$, $t \in R$ Then at $x = 0$, find f(x) and discuss continuity.

Sol. As, $y = t^2 + t\,|t|$ and $x = 2t - |t|$

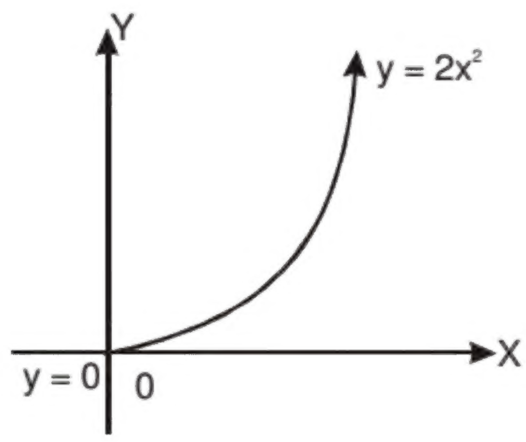
Thus when $t \ge 0$

$\Rightarrow \quad x = 2t - t = t,\ y = t^2 + t^2 = 2t^2 \quad \therefore\ x = t$ and $y = 2t^2$

$\Rightarrow \quad y = 2x^2 \ \forall\ x \ge 0$

again when, $\quad t < 0$

$\Rightarrow \quad x = 2t + t = 3t$ and $y = t^2 - t^2 = 0 \Rightarrow y = 0$ for all $x < 0$.



Hence, $f(x) = \begin{cases} 2x^2, & x \ge 0 \\ 0, & x < 0 \end{cases}$ which is clearly continuous for all x as show graphically.

Ex.14 Given the function, $f(x) = x\left[\frac{1}{x(1+x)}+\frac{1}{(1+x)(1+2x)}+\frac{1}{(1+2x)(1+3x)}+......\text{upto}\infty\right]$.

Find $f(0)$ if $f(x)$ is continuous at $x = 0$.

Sol. $f(x) = \frac{1}{1+x}+\frac{(1+2x)-(1+x)}{(1+x)(1+2x)}+\frac{(1+3x)-(1+2x)}{(1+2x)(1+3x)} + ... + \frac{(1+nx)-\left(1+\overline{n-1}x\right)}{\left(1+\overline{n-1}x\right)(1+nx)}$

$f(x) = \frac{2}{1+x} - \frac{1}{1+nx}$ upto n terms when $x \neq 0$. Hence $f(x) = \begin{cases} \frac{2}{1+x} & \text{if } x \neq 0 \text{ and } n \to \infty \\ 2 & \text{if } x = 0 \text{ for continuity.} \end{cases}$

Ex.15 Let $f : R \to R$ be a function which satisfies $f(x + y^3) = f(x) + (f(y))^3 \ \forall\ x, y \in R$. If f is continuous at x = 0, prove that f is continuous every where.

Sol. To prove $\lim\limits_{h\to 0} f(x + h) = f(x)$.

Put x = y = 0 in the given relation $f(0) = f(0) + (f(0))^3 \Rightarrow f(0) = 0$

Since f is continuous at x = 0, $\lim\limits_{h\to 0} f(h) = f(0) = 0$.

Now, $\lim\limits_{h\to 0} f(x + h) = \lim\limits_{h\to 0} f(x) + (f(h))^3 = f(x) + \lim\limits_{h\to 0} (f(h))^3 = f(x) + 0 = f(x)$.

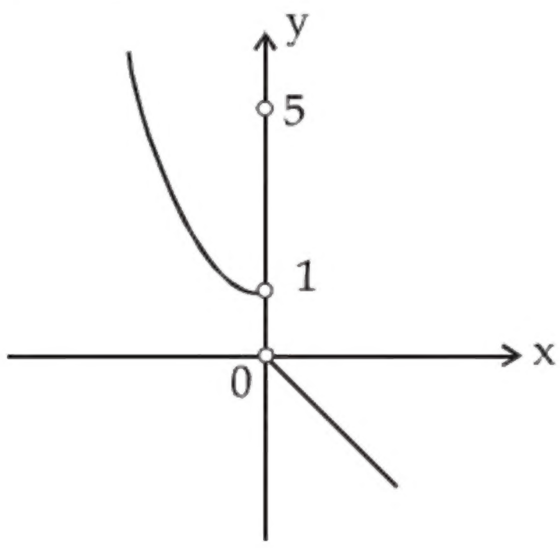
Hence f is continuous for all $x \in R$.

B. CLASSIFICATION OF DISCONTINUITY

Definition :– Let a function f be defined in the neighbourhood of a point c, except perhaps at c itself. Also let both one–sided limits $\lim\limits_{x\to c^-} f(x)$ and $\lim\limits_{x\to c^+} f(x)$ exist, where $\lim\limits_{x\to c^+} f(x) \neq \lim\limits_{x\to c^-} f(x)$.

Then the point c is called a **discontinuity of the first kind** in the function f(x).

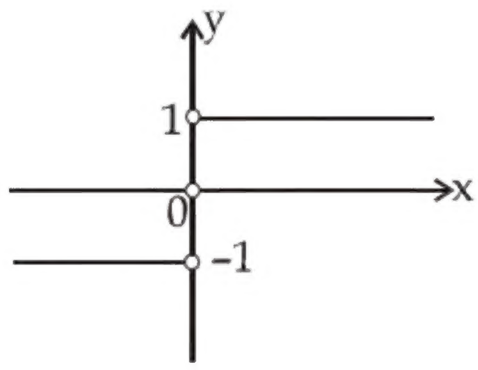
In more complicated case $\lim\limits_{x\to c} f(x)$ may not exist because one or both one-sided limits do not exist. Such condition is called a **discontinuity of the second kind.**



The function $y = \begin{cases} x^2+1 & \text{for } x<0, \\ 5 & \text{for } x=0, \\ -x & \text{for } x>0, \end{cases}$

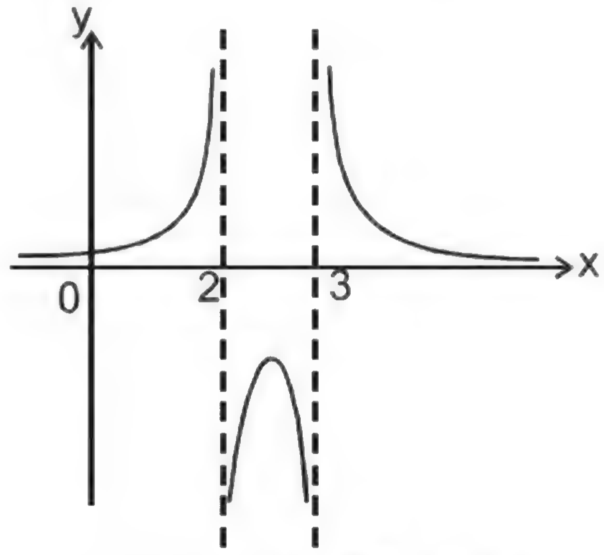
has a discontinuity of the first kind at x = 0

The function y = |x| /x is defined for all $x \in$ **R**, $x \neq 0$; but at x = 0 it has a discontinuity of the first kind.



The left-hand limit is $\lim\limits_{x\to 0^-} y = -1$, while the

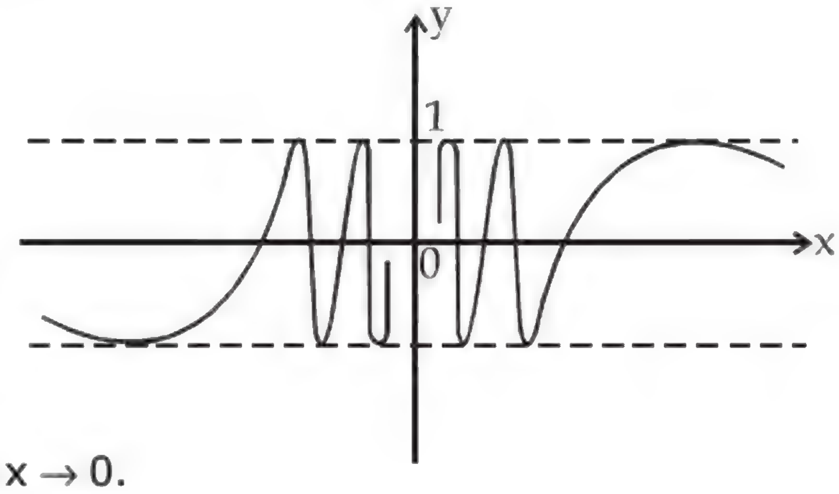
right-hand limit is $\lim\limits_{x\to 0^+} y = 1$



The function $y = \dfrac{1}{(x-2)(x-3)}$ has no limits (neither one-sided nor two-sided) at x = 2 and x = 3 since $\lim\limits_{x\to 0}\dfrac{1}{(x-2)(x-3)} = \infty$. Therefore x = 2 and x = 3 are discontinuities of the second kind

The function y = ln |x| at the point x = 0 has the limits $\lim\limits_{x\to 0} \ln|x| = -\infty$. Consequently, $\lim\limits_{x\to 0} f(x)$ (and also the one-sided limits) do not exist; x = 0 is a discontinuity of the second kind.

It is not true that discontinuities of the second kind only arise when $\lim\limits_{x\to c} f(x) = \infty$. The situation is more complicated.



Thus, the function y = sin (1/x), has no one-sided limits for $x \to 0^-$ and $x \to 0^+$, and does not tend to infinity as $x \to 0$ There is no limit as $x \to 0$ since the values of the function sin (1/x) do not approach a certain number, but repeat an infinite number of times within the interval from –1 to 1 as $x \to 0$.

Removable & Irremovable Discontinuity

(a) In case $\underset{x\to c}{\text{Limit}}$ f(x) exists but is not equal to f(c) then the function is said to have a removable discontinuity. In this case we can redefine the function such that $\underset{x\to c}{\text{Limit}}$ f(x) = f(c) & make it continuous at x = c.

Removable Type Of Discontinuity Can Be Further Classified As :

(i) Missing Point Discontinuity : where $\underset{x\to a}{\text{Limit}}$ f(x) exists finitely but f(a) is not defined . e.g. $f(x) = \dfrac{(1-x)(9-x^2)}{(1-x)}$ has a missing point discontinuity at x = 1 .

(ii) Isolated Point Discontinuity : where $\underset{x\to a}{\text{Limit}}$ f(x) exists & f(a) also exists but ; $\underset{x\to a}{\text{Limit}} \neq f(a)$.

e.g. $f(x) = \dfrac{x^2-16}{x-4}$, $x \neq 4$ & f (4) = 9 has a break at x = 4.

(b) In case $\underset{x\to c}{\text{Limit}}$ f(x) does not exist then it is not possible to make the function continuous by redefining it . Such discontinuities are known as non - removable discontinuity.

Irremovable Type Of Discontinuity Can Be Further Classified As :

(i) Finite discontinuity : e.g. $f(x) = x - [x]$ at all integral x.

(ii) Infinite discontinuity : e.g. $f(x) = \frac{1}{x-4}$ or $g(x) = \frac{1}{(x-4)^2}$ at $x = 4$.

(iii) Oscillatory discontinuity : e.g. $f(x) = \sin\frac{1}{x}$ at $x = 0$.

In all these cases the value of f(a) of the function at x= a (point of discontinuity) may or may not exist but $\underset{x\to a}{\text{Limit}}$ does not exist.

Remark :

(i) In case of finite discontinuity the non-negative difference between the value of the RHL at x = c & LHL at x = c is called **THE JUMP OF DISCONTINUITY**. A function having a finite number of jumps in a given interval I is called a **PIECE-WISE CONTINUOUS** OR **SECTIONALLY CONTINUOUS** function in this interval.

(ii) All Polynomials, Trigonometrical functions, Exponential & Logarithmic functions are continuous in their domains.

(iii) Point functions are to be treated as discontinuous . eg . $f(x) = \sqrt{1-x}+\sqrt{x-1}$ is not continuous at $x = 1$.

(iv) If f is continuous at x = c & g is continuous at x = f(c) then the composite g[f(x)] is continuous at x = c. eg . $f(x) = \frac{x\sin x}{x^2+2}$ & $g(x) = |x|$ are continuous at x = 0 , hence the composite gof (x) $= \left|\frac{x\sin x}{x^2+2}\right|$ will also be continuous at x = 0.

C. THEOREMS OF CONTINUITY

THEOREM–1 If f & g are two functions that are continuous at x= c then the functions defined by $F_1(x) = f(x) \pm g(x)$; $F_2(x) = K f(x)$, K any real number ; $F_3(x) = f(x).g(x)$ are also continuous at x= c. Further, if g (c) is not zero, then $F_4(x) = \frac{f(x)}{g(x)}$ is also continuous at x= c.

THEOREM–2 If f(x) is continuous & g(x) is discontinuous at x = a then the product function $\phi(x) = f(x) . g(x)$ is not necessarily discontinuous at x = a . e.g. $f(x) = x$ & $g(x) = \begin{bmatrix} \sin\frac{\pi}{x} & x \neq 0 \\ 0 & x = 0 \end{bmatrix}$

THEOREM–3 If f(x) and g(x) both are discontinuous at x = a then the product function $\phi(x) = f(x) . g(x)$ is not necessarily discontinuous at x = a . e.g. $f(x) = -g(x) = \begin{bmatrix} 1 & x \geq 0 \\ -1 & x < 0 \end{bmatrix}$

THEOREMS–4 : INTERMEDIATE VALUE THEOREM

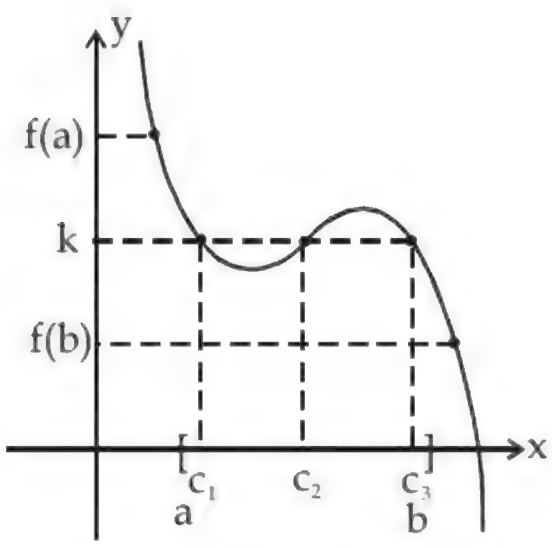
If f is continuous on the closed interval [a, b] and k is any number between f(a) and f(b), then there is at least one number c in [a, b] such that f(c) = k.

Note that the Intermediate Value Theorem tells that at least one c exists, but it does not give a method for finding c. Such theorems are called **existence theorems.**

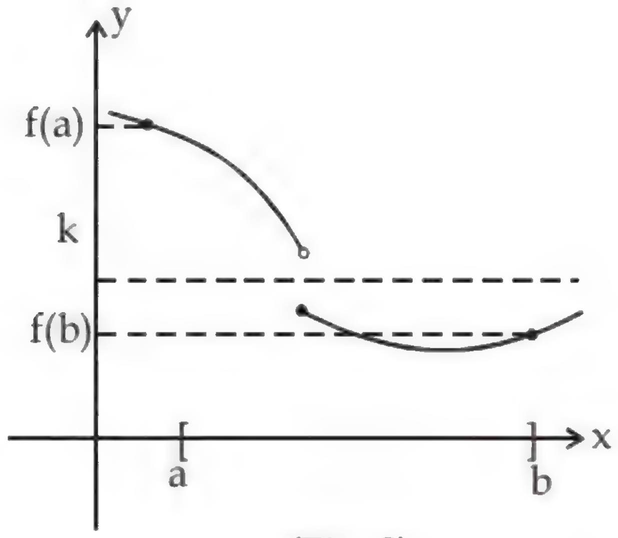
As a simple example of this theorem, consider a person's height. Suppose that a girl is 5 feet tall on her thirteenth birthday and 5 feet 7 inches tall on her fourteenth birthday. Then, for any height h between 5 feet and 7 inches, there must have been a time t when her height was exactly h. This seems reasonable because human growth is continuous and a person's height does not abruptly change from one value to another.

The Intermediate Value Theorem guarantees the existence of at least one number c in the closed interval [a, b]. There may, of course, be more than one number c such that f(c) = k, as shown in Figure 1. A function that is not continuous does not necessarily possess the intermediate value property. For example, the graph of the function shown in Figure 2 jumps over the horizontal line given by y = k and for this function there is no value of c in [a, b] such that f(c) = k.

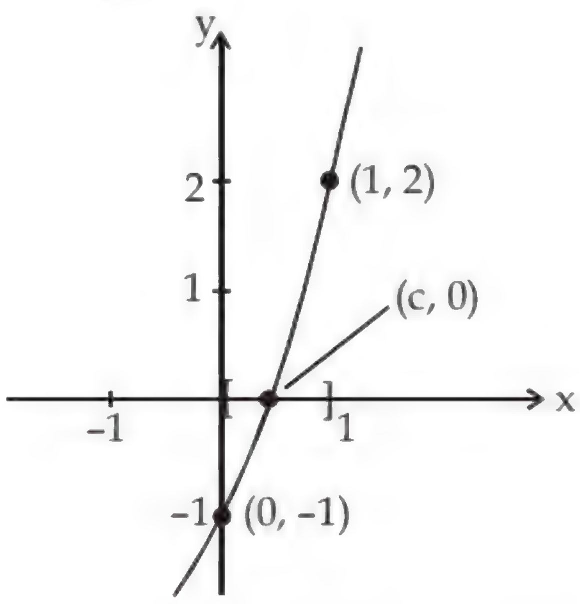
The Intermediate Value Theorem often can be used to locate the zeroes of a function that is continuous on a closed interval. Specifically, if f is continuous on [a, b] and f(a) and f(b) differ in sign, then the intermediate Value Theorem guarantees the existence of at least one zero of f in the closed interval [a, b].



(Fig. 1)
f is continuous on [a, b]. (For k, there exist 3 c's.)



(Fig. 2)
f is not continuous on [a, b]. (For k, there are no c's.)



(Fig. 3)
f is continuous on [0, 1] with f(0) < 0 and f(1) > 0.

Ex.16 Use the Intermediate Value Theorem to show that the polynomial function $f(x) = x^3 + 2x - 1$ has a zero in the interval [0, 1]

Sol. Note that f is continuous on the closed interval [0, 1]. Because

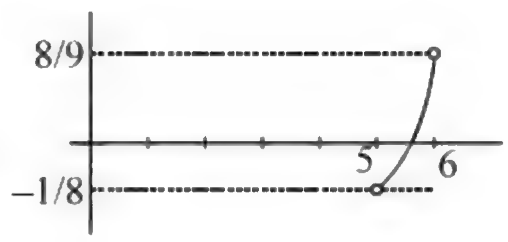
$f(0) = 0^3 + 2(0) - 1 = -1$ and $f(1) = 1^3 + 2(1) - 1 = 2$

it follows that f(0) < 0 and f(1) > 0. You can therefore apply the Intermediate Value Theorem to conclude that there must be some c in [0, 1] such that f(c) = 0, as shown in Figure 3.

Ex.17 State intermediate value theorem and use it to prove that the equation $\sqrt{x-5} = \frac{1}{x+3}$ has at least one real root.

Sol. Let $f(x) = \sqrt{x-5} - \frac{1}{x+3}$ first, $f(x)$ is continuous on [5, 6]

Also $f(5) = 0 - \frac{1}{5+3} = -\frac{1}{8} < 0$, $f(6) = 1 - \frac{1}{9} = \frac{8}{9} > 0$



Hence by intermediate value theorem $\exists$ at least one value of $c \in (5, 6)$ for which $f(c) = 0$

$\therefore \sqrt{c-5} - \frac{1}{c+3} = 0$ $\therefore$ c is root of the equation $\sqrt{x-5} = \frac{1}{x+3}$ and $c \in (5, 6)$

Ex.18 Let $f : R \to R$ satisfy $f(x) - f(y) = e^{x-y} - 1$ $\forall$ $x, y \in R$. Prove that f is a continuous function. Also prove that the function f(x) has atleast one zero if $f(0) < 1$.

Sol. $\lim\limits_{h\to 0} f(x + h) - f(x) = \lim\limits_{h\to 0} (e^{x+h-x} - 1) = 0$. Hence f is continuous everywhere.

Now $f(x) = f(0) + e^x - 1 \Rightarrow \lim\limits_{x\to\infty} f(x) = \infty$ $\lim\limits_{x\to-\infty} f(x) = f(0) - 1 < 0$

Since f(x) is positive for large positive x and negative of large negative x, by Intermediate Value Theorem f(x) = 0 has atleast one root.

Ex.19 If f(x) be a continuous function in $[0, 2\pi]$ and $f(0) = f(2\pi)$ then prove that there exists point $c \in (0, \pi)$ such that $f(c) = f(c + \pi)$.

Sol. Let $g(x) = f(x) - f(x + \pi)$ ....(i)

at $x = \pi$; $g(\pi) = f(\pi) - f(2\pi)$ ....(ii)

at $x = 0$, $g(0) = f(0) - f(\pi)$ ...(iii)

adding (ii) and (iii), $g(0) + g(\pi) = f(0) - f(2\pi)$

$\Rightarrow$ $g(0) + g(\pi) = 0$ [Given $f(0) = f(2\pi)$ $\Rightarrow$ $g(0) = -g(\pi)$

$\Rightarrow$ $g(0)$ and $g(\pi)$ are opposite in sign.

$\Rightarrow$ There exists a point c between 0 and π such $g(c) = 0$ as shown in graph;

From (i) putting $x = c$ $g(c) = f(c) - f(c + \pi) = 0$ Hence, $f(c) = f(c + \pi)$

D. DIFFERENTIABILITY

Definition of Tangent : If f is defined on an open interval containing c, and if the limit

$$\lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x} = m$$

exists, then the line passing through (c, f(c)) with slope m is the tangent line to the graph of f at the point (c, f(c)).

The slope of the tangent line to the graph of f at the point (c, f(c)) is also called the slope of the graph of f at x = c.

The above definition of a tangent line to a curve does not cover the possibility of a vertical tangent line. For vertical tangent lines, you can use the following definition. If f is continuous at c and

$$\lim_{\Delta x \to 0}\left|\frac{f(c+\Delta x)-f(c)}{\Delta x}\right| = \infty$$

then the vertical line, x = c, passing through (c, f(c)) is a vertical tangent line to the graph of f. For example, the function shown in Figure has a vertical tangent line at (c, f(c)). If the domain of f is the closed interval [a, b], then you can extend the definition of a vertical tangent line to include the endpoints by considering continuity and limits from the right (for x = a) and from the left (for x = b).

$$\lim_{\Delta x \to 0^+}\left|\frac{f(a+\Delta x)-f(a)}{\Delta x}\right| = \infty$$

$$\lim_{\Delta x \to 0^-}\left|\frac{f(b+\Delta x)-f(b)}{\Delta x}\right| = \infty$$

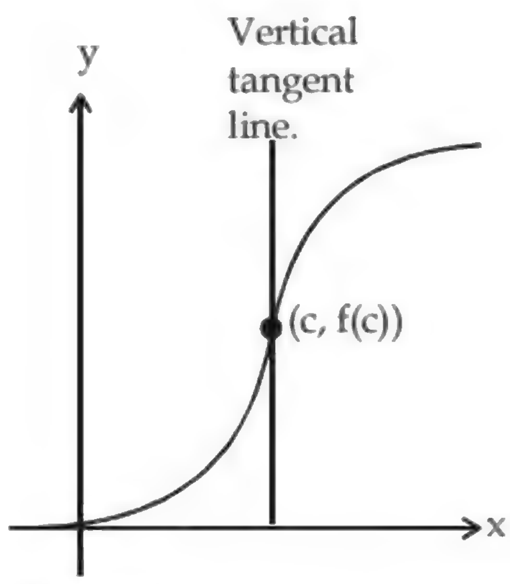


Figure
The graph of f has a vertical tanent line at (c, f(c)).

In the preceding section we considered the derivative of a function f at a fixed number a :

$$f'(a) = \lim_{h \to 0}\frac{f(a+h)-f(a)}{h} \qquad ...(1)$$

Note that alternatively, we can define

$f¢(a) = \underset{x \to a}{\text{Limit}}\ \frac{f(x)-f(a)}{x-a}$, provided the limit exists.

Here we change our point of view and let the number a vary. If we replace a in Equation 1 by a variable

x, we obtain $f'(x) = \lim_{h \to 0}\frac{f(x+h)-f(x)}{h}$ ...(2)

Given any number x for which this limit exists, we assign to x the number f′(x). So we can regard f′ as a new function, called the **derivative of** f and defined by Equation 2. We know that the value of f′(x), can be interpreted geometrically as the slope of the tangent line to the graph of f at the point (x, f(x)).

The function f′ is called the derivative of f because it has been "derived" from f by the limiting operation in Equation 2. The domain of f′ is the set {x|f′(x) exists} and may be smaller than the domain of f.

Average And Instantaneous Rate Of Change

Suppose y is a function of x, say y = f(x). Corresponding to a change from x to x + Δx, the variable y changes from f(x) to f(x + Δx). The change in y is $\Delta y = f(x + \Delta x) - f(x)$, and the **average rate of change of y with respect to x** is

$$\text{Average rate of change} = \frac{\text{change in y}}{\text{change in x}} = \frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

As the interval over which we are averaging becomes shorter (that is, as $\Delta x \to 0$), the average rate of change approaches what we would intuitively call the **instantaneous rate of change of y with respect to x,** and the difference quotient approaches the derivative $\frac{dy}{dx}$. Thus, we have

$$\text{Instantaneous Rate of Change} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = f'(x)$$

To summarize :

Instantaneous Rate of Change

Suppopse f(x) is differentiable at x = x_0. Then the **instantaneous rate of cange** of y = f(x) with respect to x at x_0 is the value of the derivative of f at x_0. That is

$$\text{Instantaneous Rate of Change} = f'(x_0) = \left.\frac{dy}{dx}\right|_{x=x_0}$$

Ex.20 Find the rate at which the function $y = x^2 \sin x$ is changing with respect to x when x = π.

For any x, the instantaneous rate of change in the derivative,

Sol. $\frac{dy}{dx} = 2x \sin x + x^2 \cos x$

Thus, the rate when x = π is $\left.\frac{dy}{dx}\right|_{x=\pi} = 2\pi \sin \pi + \pi^2 \cos \pi = 2\pi(0) + \pi^2 (-1) = -\pi^2$

The negative sign indicates that when x = π, the function is decreasing at the rate of $\pi^2 \approx 9.9$ units of y for each one-unit increase in x.

Let us consider an example comparing the average rate of change and the instantaneous rate of change.

Ex.21 Let $f(x) = x^2 - 4x + 7$.

(a) Find the instantaneous rate of change of f at x = 3.

(b) Find the average rate of change of f with respect to x between x = 3 and 5.

Sol. **(a)** The derivative of the function is $f'(x) = 2x - 4$

Thus, the instantaneous rate of change of f

at x = 3 is $f'(3) = 2(3) - 4 = 2$

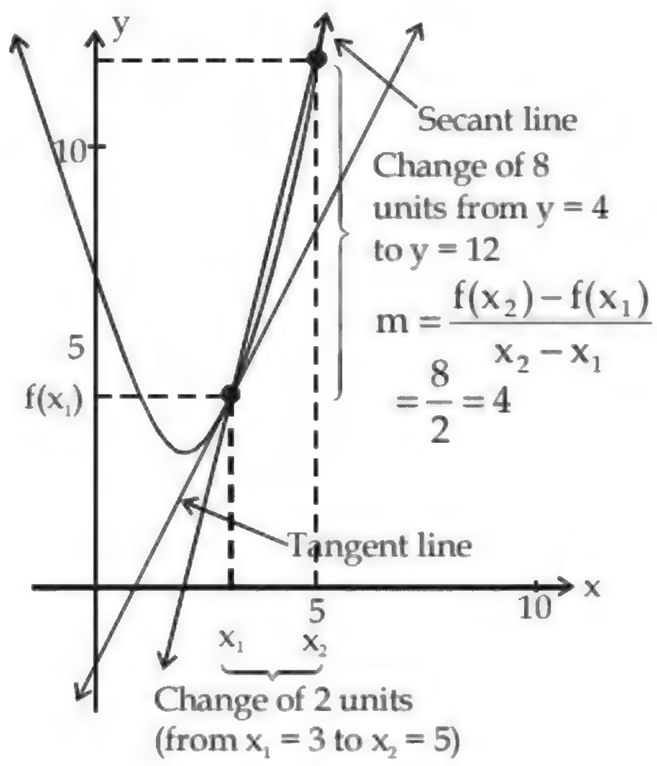
The tangent line at x = 3 has slope 2, as shown in the figure.

(b) The (average) rate of change from x = 3 to x = 5 is found by dividing the change in f by the change in x. The change in f from x = 3 to x = 5 is

$f(5) - f(3) = [5^2 - 4(5) + 7] - [3^2 - 4(3) + 7] = 8$

Thus, the average rate of change is $\frac{f(5) - f(3)}{5.3} = \frac{8}{2} = 4$

The slope of the secant line is 4, as shown in the figure.



Figure

E. RELATION BETWEEN CONTINUITY & DIFFERENTIABILITY

If a function f is derivable at x then f is continuous at x.

For : $f'(x) = \underset{h\to 0}{\text{Limit}} \frac{f(x+h)-f(x)}{h}$ exists.

Also $f(x+h)-f(x) = \frac{f(x+h)-f(x)}{h}.h \; [h \neq 0]$

Therefore $\underset{h\to 0}{\text{Limit}} [f(x+h)-f(x)] = \underset{h\to 0}{\text{Limit}} \frac{f(x+h)-f(x)}{h}.h = f'(x).0 = 0$

Therefore $\underset{h\to 0}{\text{Limit}} [f(x+h)-f(x)] = 0 \Rightarrow \underset{h\to 0}{\text{Limit}} f(x+h) = f(x) \Rightarrow$ f is continuous at x.

If f(x) is derivable for every point of its domain, then it is continuous in that domain .

The converse of the above result is not true :

"If f is continuous at x, then f may or maynot be derivable at x"

The functions $f(x) = |x|$ & $g(x) = x \sin \frac{1}{x}$; $x \neq 0$ & $g(0) = 0$ are continuous at x = 0 but not derivable at x = 0.

Remark :

(a) Let $f'_{+}(a) = p$ & $f'_{-}(a) = q$ where p & q are finite then :

(i) $p = q \Rightarrow$ f is derivable at $x = a$ $\Rightarrow$ f is continuous at $x = a$.

(ii) $p \neq q \Rightarrow$ f is not derivable at $x = a$ but f is continuous at $x = a$

Differentiable $\Rightarrow$ Continuous ; Non-differentiable $\not\Rightarrow$ Discontinuous

But Discontinuous $\Rightarrow$ Non-differentiable .

(b) If a function f is not differentiable but is continuous at $x = a$ it geometrically implies a sharp corner at $x = a$.

Ex.22 Given $f(x) = x^2 . \text{sgn}(x)$ examine the continuity and derivability at the origin.

Sol. $f(x) = \begin{cases} x^2 & \text{if } x>0 \\ 0 & \text{if } x=0 \\ -x^2 & \text{if } x<0 \end{cases}$ $\quad f'(0^+) = \underset{h\to 0}{\text{Limit}} \dfrac{h^2-0}{h} = 0$

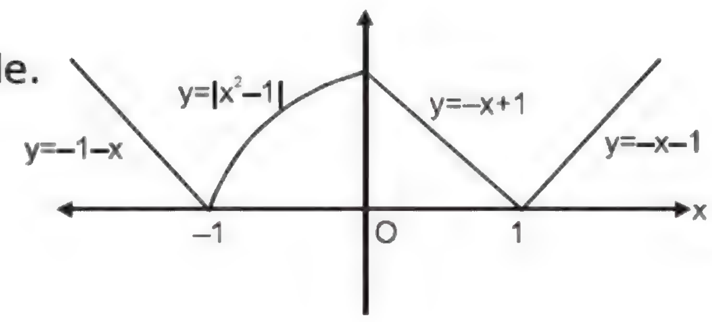
$f'(0^-) = \underset{h\to 0}{\text{Limit}} \; - \dfrac{h^2-0}{-h} = 0 \Rightarrow$ f is derivable at $x = 0$ $\Rightarrow$ continuous at $x = 0$

Ex.23 If $f(x) = \begin{cases} -1-x & ; \; x \le -1 \\ |x^2-1| & ; \; -1 < x \le 0 \\ k(-x+1) & ; \; 0 < x \le 1 \\ |x-1| & ; \; x > 1 \end{cases}$, then find the value of k so that f(x) becomes continuous at $x = 0$.

Hence, find all the points where the functions is non-differentiable.

Sol. From the graph of f(x) it is clear that for the function to be continuous only possible value of k is 1.

Points of non-differentiability are $x = 0, \pm 1$.



Ex.24 Examine the function , $f(x) = x . \dfrac{a^{1/x}-a^{-1/x}}{a^{1/x}+a^{-1/2}}$, $x \neq 0$ $(a > 0)$ and $f(0) = 0$ for continuity and existence of the derivative at the origin.

Sol. If $a \in (0, 1)$ $\quad f'(0^+) = -1$; $f'(0^-) = 1$ $\Rightarrow$ continuous but not derivable

$a = 1$; $f(x) = 0$ which is constant $\Rightarrow$ continuous and derivable

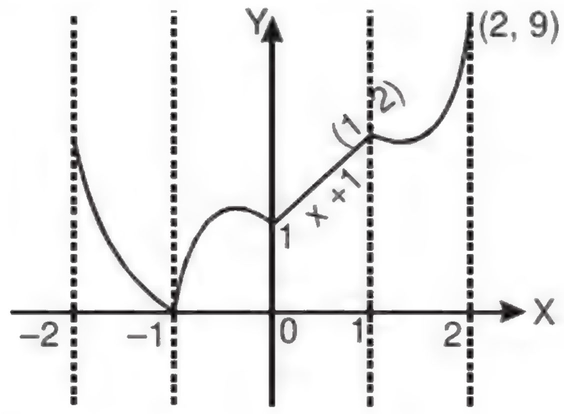
If $a > 1$ $\quad f'(0^-) = -1$; $f'(0^+) = 1$ $\Rightarrow$ continuous but not derivable

Ex.25 If f(x) = |x + 1| {|x| + |x − 1|}, then draw the graph of f(x) in the interval [−2, 2] and discuss the continuity and differentiablity in [−2, 2]

Sol. Here, f(x) = |x + 1| { |x| + |x − 1| }

$$f(x) = \begin{cases} (x+1)(2x-1); & -2 \le x < -1 \\ -(x+1)(2x-1); & -1 \le x < 0 \\ (x+1); & 0 \le x < 1 \\ (x+1)(2x-1); & 1 \le x \le 2 \end{cases}$$

Thus the graph of f(x) is;



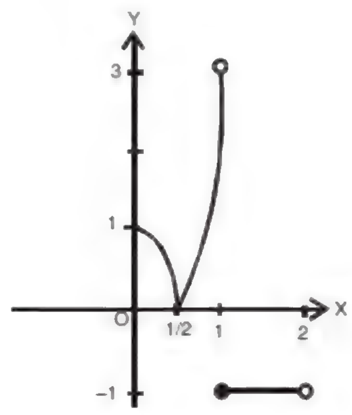
which is clearly, continuous for $x \in R$ and, differentiablity for $x \in R - \{-1, 0, 1\}$

Ex.26 If $f(x) = \begin{cases} |1-4x^2| & , 0 \le x < 1 \\ |x^2-2x| & , 1 \le x < 2 \end{cases}$ where [.] denotes the greatest integer function.

Discuss the continuity and differentiability of f(x) in [0, 2).

Sol. Since $1 \le x < 2 \Rightarrow 0 \le x - 1 < 1$ then $[x^2 - 2x] = [(x-1)^2 - 1] = [(x-1)^2] - 1 = 0 - 1 = -1$

$$\therefore \quad f(x) = \begin{cases} 1-4x^2 & , 0 \le x < \frac{1}{2} \\ 4x^2-1 & , \frac{1}{2} \le x < 1 \\ -1 & , 1 \le x < 2 \end{cases} \qquad \therefore \quad \text{Graph of } f(x) :$$



It is clear from the graph that f(x) is discontinuous at x = 1 and not differentiable at $x = \frac{1}{2}$ and x = 1.

Further details are as follows : $f(x) = \begin{cases} 1-4x^2, & 0 \le x < 1/2 \\ 4x^2-1, & 1/2 \le x < 1 \\ -1, & 1 \le x < 2 \end{cases} \Rightarrow f'(x) = \begin{cases} -8x, & 0 \le x < 1/2 \\ 8x, & 1/2 \le x < 1 \\ 0, & 1 \le x < 2 \end{cases}$

$$\Rightarrow \quad f'(x) = \begin{cases} -4 & x < 1/2 \\ 4 & x > 1/2 \end{cases} \quad \text{and} \quad f'(x) = \begin{cases} 8, & x < 1 \\ 0, & x > 1 \end{cases}$$

Hence, which shows f(x) is not differentiable at x = 1/2 (as RHD = 4 and LHD = −4) and x = 1 (as RHD = 0 and LHD = 8). Therefore, f(x) is differentiable, $\forall\ x \in [0, 2) - \{1/2, 1\}$

Ex.27 Let $f(x) = \left[\begin{array}{ll} 2x^2 \sin \pi x & x \le 1 \\ x^3 + ax^2 + b & x > 1 \end{array} \right.$ be a differential function . Examine whether it is twice differentiable in R.

Sol. differentiability of f gives : a + b + 1 = 0 and $\left. \begin{array}{l} f'(1^+) = 3+2a \\ f'(1^-) = -2\pi \end{array} \right] \Rightarrow a = -\frac{2\pi+3}{2}$ and $b = \frac{2\pi+3}{2}$

Thus $f'(x) = \begin{cases} 4x\sin\pi x + 2\pi x^2\cos\pi x, & x \le 1 \\ 3x^2 - (3+2\pi)x, & x > 1 \end{cases}$ & $f''(x) = \begin{cases} (4-2\pi^2x^2)\sin\pi x + 8\pi x\cos\pi x, & x < 1 \\ 6x - 3 - 2\pi, & x > 1 \end{cases}$

$f''(1^-) = -8\pi$ & $f''(1^+) = 3 - 2\pi$ $\Rightarrow$ $f''(x)$ is discontinuous at $x = 1$

Hence $f(x)$ is twice differentiable for x except at $x = 1$]

Ex.28 Suppose $f(x) = \begin{cases} x^3 & \text{if } x < 1 \\ ax^2 + bx + c & \text{if } x \ge 1 \end{cases}$. If $f''(1)$ exist then find the value of $a^2 + b^2 + c^2$.

Sol. For continuity at x = 1 we leave $f(1^-) = 1$ and $f(1^+) = a + b + c$

$\therefore$ $a + b + c = 1$ ....(1)

$f'(x) = \begin{cases} 3x^2 & \text{if } x < 1 \\ 2ax + b & \text{if } x \ge 1 \end{cases}$ for continuity of $f'(x)$ at $x = 1$ $f'(1^-) = 3$; $f'(1^+) = 2a + b$

hence $2a + b = 3$ ....(2)

$f''(x) = \begin{cases} 6x & \text{if } x < 1 \\ 2a & \text{if } x \ge 1 \end{cases}$ $f''(1^-) = 6$; $f''(1^+) = 2a$ for continuity of $f''(x)$ $2a = 6 \Rightarrow a = 3$

from (2), $b = -3$; $c = 1$. Hence $a = 3$, $b = -3$; $c = 1$ $\therefore$ $\sum a^2 = 19$

Ex.29 Check the differentiability of the function $f(x) = \max\{\sin^{-1}|\sin x|, \cos^{-1}|\sin x|\}$.

Sol. $\sin^{-1}|\sin x|$ is periodic with period π $\Rightarrow$ $\sin^{-1}|\sin x| = \begin{cases} x, & n\pi \le x \le n\pi + \frac{\pi}{2} \\ \pi - x, & n\pi + \frac{\pi}{2} \le x \le n\pi + \pi \end{cases}$

Also $\cos^{-1}|\sin x| = \frac{\pi}{2} - \sin^{-1}|\sin x|$

$\Rightarrow$ $f(x) = \max\begin{cases} x, \frac{\pi}{2} - x, & n\pi \le x \le n\pi + \frac{\pi}{2} \\ \pi - x, x - \frac{\pi}{2}, & n\pi + \frac{\pi}{2} \le x \le n\pi + \pi \end{cases}$ $\Rightarrow$ $f(x) = \begin{cases} \frac{\pi}{2} - x, & n\pi \le x \le n\pi + \frac{\pi}{4} \\ x, & n\pi + \frac{\pi}{4} < n\pi + \frac{\pi}{2} \\ \pi - x, & n\pi + \frac{\pi}{2} < x \le n\pi + \frac{3\pi}{4} \\ x - \frac{\pi}{2}, & n\pi + \frac{3\pi}{4} < x \le n\pi + \pi \end{cases}$

$\Rightarrow$ f(x) is not differentiable at $x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$.........

$\Rightarrow$ f(x) is not differentiable at $x = \frac{n\pi}{4}$.

Ex.30 Let $f(x) = \cos^{-1}\left(\frac{2x}{1+x^2}\right)$. Define $f'(x)$ for every $x \in R$ stating clearly the point where $f(x)$ is not differentiable.

Sol. $f(x) = \cos^{-1}\left(\frac{2x}{1+x^2}\right)$, $x \in R \Rightarrow f'(x) = -\frac{1}{\sqrt{1-\left(\frac{2x}{1+x^2}\right)^2}} \cdot \frac{2(1+x^2)-4x^2}{(1+x^2)^2}$; $f'(x) = -\frac{2(1-x^2)}{|1-x^2|(1+x^2)}$

Since $\lim\limits_{x\to1^-} f'(x)$ and $\lim\limits_{x\to1^+} f'(x)$ have finite values which are unequal, $f'(1)$ does not exist.

Similarly $f'(-1)$ does not exist. hence $f'(x) = \begin{cases} -\frac{2}{1+x^2} & \text{if } -1<x<1 \\ \text{non existent} & \text{if } x=\pm1 \\ \frac{2}{1+x^2} & \text{if } x>1 \text{ or } x<-1 \end{cases}$

Ex.31 Find the interval of values of k for which the function $f(x) = |x^2 + (k-1)|x| - k|$ is non differentiable at five points.

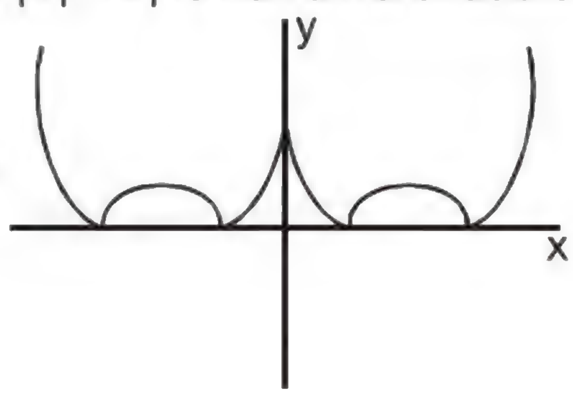
Sol. $f(x) = |x^2 + (k-1)|x| - k| = |(|x| - 1)(|x| + k)|$

Also $f(x)$ is an even function and $f(x)$ is not differentiable at five points.

So $|(x-1)(x+k)|$ is non differentiable for two positive values of x.

$\Rightarrow$ Both the roots of $(x-1)(x+k) = 0$ are positive.

$\Rightarrow$ $k < 0 \Rightarrow$ $k \in (-\infty, 0)$.



Ex.32 Let $f(x) = \begin{cases} \left(\dfrac{(1+\{x\})\frac{1}{\{x\}}}{e}\right)^{\frac{1}{\{x\}}}, & x \neq \text{Integer} \\ \dfrac{2}{e}, & x = \text{Integer} \end{cases}$. Discuss the continuity and differentiability of $f(x)$ at any integral point. (where $\{*\}$ denotes the fractional part)

TRY TO SOLVE THIS QUESTION YOURSELF.

$$f(I_0^-) = \lim_{h\to 0} f(I_0 - h) = \lim_{h\to 0}\left(\frac{(1+\{I_0-h\})^{\frac{1}{\{I_0-h\}}}}{e}\right) = \left(\frac{(1+1)^1}{e}\right)^1 = \frac{2}{e}$$

$$f(I_0 + h) = \lim_{h\to 0}(I_0 + h) = \lim_{h\to 0}\left(\frac{(1+h)^{1/h}}{e}\right)^{1/h} = e^{\lim_{h\to 0}\left(\frac{(1+h)^{1/h}}{e}\right)^{1/h}} = e^{\lim_{h\to 0}\frac{1}{h}\left(\frac{(1+h)^{1/h}}{e}-1\right)} = e^{\lim_{h\to 0}\frac{1}{h}\left(\frac{(1+h)^{1/h}}{e}-1\right)} = e^{\lim_{h\to 0}\frac{(1+h)^{1/h}-e}{eh}}$$

$$\text{Now } (1+h)^{1/h} = e^{\frac{1}{h}\ln(1+h)} = e^{\frac{1}{h}\left(h-\frac{h^2}{2}+\frac{h^3}{3}-\dots\right)} = e.e^{-h\left(\frac{1}{2}-\frac{h}{3}+\frac{h^2}{4}-\dots\right)}$$

$$\Rightarrow\quad f(I_0^+) = e^{\lim_{h\to 0}\frac{e^{-h\left(\frac{1}{2}-\frac{h}{3}+\frac{h^2}{4}-\dots\right)}-1}{h}} = e^{\lim_{h\to 0}\frac{\left(\frac{e^{-h\left(\frac{1}{2}-\frac{h}{3}+\frac{h^2}{4}-\dots\right)}-1}{-h\left(\frac{1}{2}-\frac{h}{3}+\frac{h^2}{4}-\dots\right)}\right)\left(\frac{1}{2}-\frac{h}{3}+\frac{h^2}{4}-\dots\right)(-1)}{}} = e^{-\frac{1}{2}}$$

Since $f(I_0 + 0) \neq f(I_0 - 0) \Rightarrow f(x)$ is discontinuous at any integeral point and hence non-differentiable.

Definition : A function f is **differentiable at a** if f′(a) exists. It is **differentiable on an open interval** (a,b) [or (a, ∞) or (−∞, a) or (− ∞, ∞)] if it is differentiable at every number in the interval.

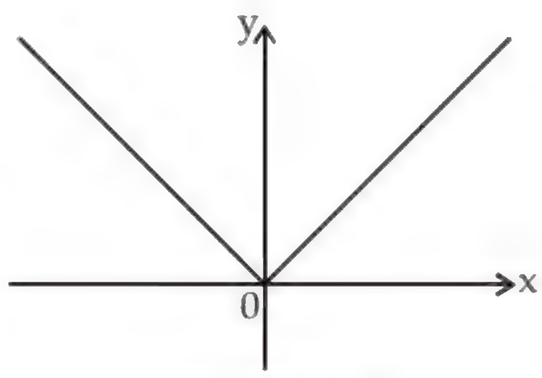
Derivability Over An Interval : f(x) is said to be derivable over an interval if it is derivable at each & every point of the interval. f(x) is said to be derivable over the closed interval [a, b] if :

(i) for the points a and b, f ′(a+) & f ′(b−) exist &

(ii) for any point c such that a < c < b, f ′(c+) & f′(c−) exist & are equal .

How Can a Function Fail to Be Differentiable ?

We see that the function y = |x| is not differentiable at 0 and Figure shows that its graph changes direction abruptly when x = 0. In general, if the graph of a function f has a "corner" or "kink" in it, then the graph of f has no tangent at this point and f is not differentiable there. [In trying to compute f ′(a), we find that the left and right limits are different.]

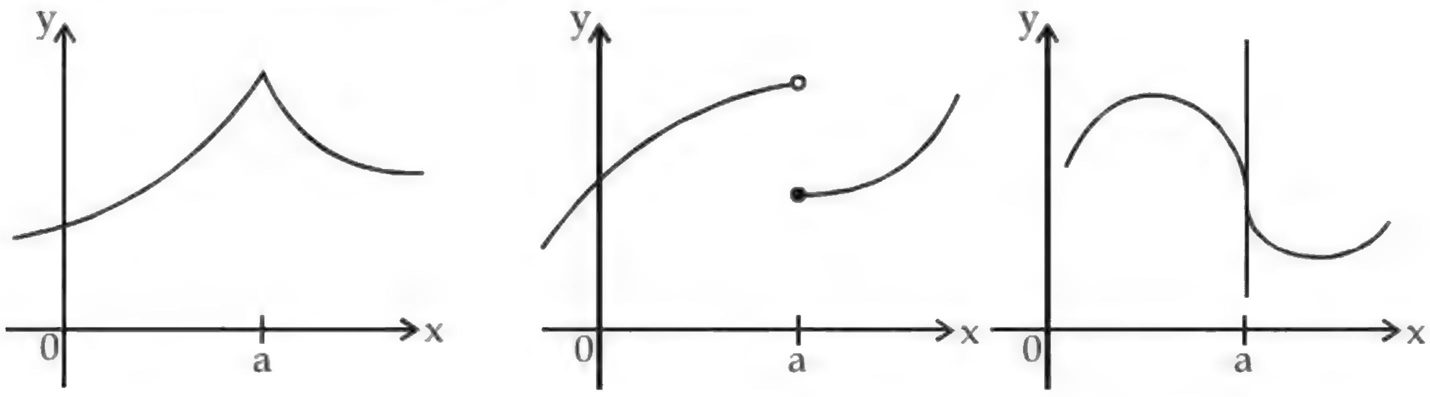


y = f(x) = | x |
Figure

There is another way for a function not to have a derivative. If f is discontinuous at a, then f is not differentiable at a. So at any discontinuity (for instance, a jump discontinuity), f fails to be differentiable.

A third possibility is that the curve has a **vertical tangent line** when at x = a, $\lim\limits_{x\to a} |f'(x)| = \infty$

This means that the tangent lines become steeper and steeper as $x \to a$. Figure (a, b, c) illustrates the three posibilities that we have discussed.



(a) A corner (b) A discontinuity (c) A Vertical tangent

Right hand & Left hand Derivatives By definition : $f'(a) = \text{Limit}_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$

(i) The right hand derivative of f ′ at x = a denoted by $f'_+(a)$ is defined by :

$$f'_+(a) = \text{Limit}_{h\to 0^+} \frac{f(a+h)-f(a)}{h}$$, provided the limit exists & is finite.

(ii) The left hand derivative of f at x = a denoted by $f'_-(a)$ is defined by :

$$f'_-(a) = \text{Limit}_{h\to 0^+} \frac{f(a-h)-f(a)}{-h}$$, Provided the limit exists & is finite.

We also write $f'_+(a) = f'(a^+)$ & $f'_-(a) = f'(a^-)$.
f′(a) exists if and only if these one-sided derivatives exist and are equal.

Ex.33 If a function f is defined by $f(x) = \begin{cases} \dfrac{xe^{1/x}}{1+e^{1/x}}, & x \neq 0 \\ 0 & , x = 0 \end{cases}$ show that f is continuous but not derivable at x = 0

Sol. We have $f(0+0) = \lim\limits_{x\to 0+0} \dfrac{xe^{1/x}}{1+e^{1/x}} = \lim\limits_{x\to 0+0} \dfrac{e^{1/x}}{1+e^{1/x}} = \lim\limits_{x\to 0+0} \dfrac{x}{e^{1/x}+1} = 0$

$$f(0-0) = \lim_{x\to 0-0} \frac{xe^{1/x}}{1+e^{1/x}} = 0$$

Alsof(0) = 0 ∴ f(0 + 0) = f(0 – 0) = f(0) ⇒ f is continuous at x = 0

Again $f'(0+0) = \lim\limits_{x\to 0+0} \dfrac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0+0} \dfrac{\dfrac{xe^{1/x}}{1+e^{1/x}}-0}{x} = \lim\limits_{x\to 0+0} \dfrac{e^{1/x}}{1+e^{1/x}} = \lim\limits_{x\to 0+0} \dfrac{1}{e^{-1/x}+1} = 1$

$$f'(0-0) = \lim_{x\to 0-0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0-0} \frac{\dfrac{xe^{1/x}}{1+e^{1/x}}-0}{x} = \lim_{x\to 0-0} \frac{e^{1/x}}{1+e^{1/x}} = 0$$

Since f′(0 + 0) ≠ f′(0 – 0), the derivative of f(x) at x = 0 does not exist.

Ex.34 A function f(x) is such that $f\left(x+\frac{\pi}{2}\right)=\frac{\pi}{2}$ - |x| ∀ x. Find $f'\left(\frac{\pi}{2}\right)$, if it exists.

Sol. Given that $f\left(x+\frac{\pi}{2}\right)=\frac{\pi}{2}-|x|$ $\Rightarrow$ $f'\left(\frac{\pi^+}{2}\right)=\lim\limits_{h\to 0}\frac{f\left(\frac{\pi}{2}+h\right)-f\left(\frac{\pi}{2}\right)}{h}=\frac{\frac{\pi}{2}-|h|-\frac{\pi}{2}}{h}=-1$

$\Rightarrow$ and $f'\left(\frac{\pi^-}{2}\right)=\lim\limits_{h\to 0}\frac{f\left(\frac{\pi}{2}-h\right)-f\left(\frac{\pi}{2}\right)}{-h}=\frac{\frac{\pi}{2}-|-h|-\frac{\pi}{2}}{-h}=1$ $\Rightarrow$ $f'\left(\frac{\pi}{2}\right)$ doesn't exist.

Ex.35 Let *f* be differentiable at x = a and let *f* (a) ≠ 0. Evaluate $\lim\limits_{n\to\infty}\left\{\frac{f(a+1/n)}{f(a)}\right\}^n$.

Sol. $l=\lim\limits_{n\to\infty}\left\{\frac{f(a+1/n)}{f(a)}\right\}^n$ (1^∞ form)

$l=e^{\left(\lim\limits_{n\to\infty} n\left\{\frac{f(a+1/n)-f(a)}{f(a)}\right\}\right)}=e^{\left(\lim\limits_{h\to 0}\frac{f(a+h)-f(a)}{h}\cdot\frac{1}{f(a)}\right)}=e^{\frac{f'(a)}{f(a)}}$ (put n = 1/h)

Ex.36 A function *f* is defined as , $f(x)=\begin{cases}\frac{1}{|x|} & \text{if } |x|\ge\frac{1}{2}\\ a+bx^2 & \text{if } |x|<\frac{1}{2}\end{cases}$. If *f* (x) is derivable at x = 1/2 find the values of 'a' and 'b'.

Sol. $f(x)=\begin{cases}\frac{1}{|x|} & \text{if } x\ge\frac{1}{2} \text{ or } x\le-\frac{1}{2}\\ a+bx^2 & \text{if } -\frac{1}{2}<x<\frac{1}{2}\end{cases}=\begin{cases}\frac{1}{x} & \text{if } x\ge\frac{1}{2}\\ -\frac{1}{x} & \text{if } x\le-\frac{1}{2}\\ a+bx^2 & \text{if } -\frac{1}{2}<x<\frac{1}{2}\end{cases}$

$$f'\left(\frac{1^+}{2}\right)=\lim_{h\to 0}\frac{f\left(\frac{1}{2}+h\right)-f\left(\frac{1}{2}\right)}{h}=\lim_{h\to 0}\frac{\frac{1}{\left(\frac{1}{2}+h\right)}-2}{h}=\lim_{h\to 0}\frac{1-2\left(\frac{1}{2}+h\right)}{h\left(\frac{1}{2}+h\right)}=\lim_{h\to 0}\frac{-2h}{h\left(\frac{1}{2}+h\right)}=-4$$

$$f'\left(\frac{1^-}{2}\right)=\lim_{h\to 0}\frac{f\left(\frac{1}{2}-h\right)-f\left(\frac{1}{2}\right)}{-h}=\lim_{h\to 0}\frac{a+b\left(\frac{1}{2}-h\right)^2-2}{-h}=\lim_{h\to 0}\frac{\left(a+\frac{b}{4}-2\right)-bh+h^2}{-h}$$

for existence of limit $a+\frac{b}{4}=2$ ....(1) $\Rightarrow$ $f'\left(\frac{1^-}{2}\right)$ = b

$\therefore f'\left(\frac{1^-}{2}\right) = b = f'\left(\frac{1^+}{2}\right) = -4$ $\therefore$ a – 1 = 2 $\Rightarrow$ a = 3 Hence a = 3 and b = – 4

Ex.37 Let f : R $\rightarrow$ R satisfying $|f(x)| \le x^2$, $\forall$ x $\in$ R then show f(x) is differentiable at x = 0.

Sol. Since, $|f(x)| \le x^2$, $\forall$ x $\in$ R $\therefore$ at x = 0, $|f(0)| \le 0 \Rightarrow$ f(0) = 0 ...(i)

$\therefore$ $f'(0) = \lim_{h\to0}\frac{f(h)-f(0)}{h}$ $\lim_{h\to0}\frac{f(h)}{h}$ ....(ii) {f(0) = 0 from (i)}

Now, $\left|f\frac{(h)}{h}\right| \le |h| \Rightarrow -|h| \le f\frac{(h)}{h} \le |h| \Rightarrow \lim_{h\to0} f\frac{(h)}{h} \to 0$ ...(iii) {using Cauchy–Squeeze theorem}

$\therefore$ from (ii) and (iii), we get $f'(0) = 0$. i.e. f(x) is differentiable at x = 0.

F. OPERATION ON DIFFERENTIABLE FUNCTIONS

1. If f(x) & g(x) are derivable at x = a then the functions f(x) + g(x), f(x) – g(x), f(x). g(x) will also be derivable at x = a & if g (a) $\ne$ 0 then the function f(x)/g(x) will also be derivable at x = a.

If f and g are differentiable functions, then prove that their product fg is differentiable.

Let a be a number in the domain of fg. By the definition of the product of two functions we have

(fg) (a) = f(a) g(a) (fg) (a + t) = f(a + t) g(a + t).

Hence $(fg)'(a) = \lim_{t\to0}\frac{f(g)(a+t)-(fg)(a)}{t} = \lim_{t\to0}\frac{f(a+t)g(a+t)-f(a)g(a)}{t}$

The following algebraic manipulation will enable us to put the above fraction into a form in which we can see what the limit is:

f(a + t) g(a + t) – f(a) g(a) = f(a + t) g(a + t) – f(a) g (a + t) + f(a)g(a + t) – f(a) g(a)

= [f(a + t) – f(a)] g(a + t) + [g(a + t) – g(a)] f(a).

Thus $(fg)'(a) = \lim_{t\to0}\left[\frac{f(a+t)-f(a)}{t}g(a+t)+\frac{g(a+t)-g(a)}{t}f(a)\right]$.

The limit of a sum of products is the sum of the products of the limits. Moreover, f′(a) and g′(a) exist by hypothesis. Finally, since g is differentiable at a, it is continuous there ; and so $\lim_{t\to0}$ g(a + t) = f(a). We conclude that

$$(fg)'(a) = \left[\lim_{t\to0}\frac{f(a+t)-f(a)}{t}\right]\lim_{t\to0}g(a+t) + \left[\lim_{t\to0}\frac{g(a+t)-g(a)}{t}\right]f(a)$$

= f′(a)g(a) + g′(a)f(a) = (f′g + g′f) (a).

2. If $f(x)$ is differentiable at $x = a$ & $g(x)$ is not differentiable at $x = a$, then the product function $F(x) = f(x) . g(x)$ can still be differentiable at $x = a$ e.g. $f(x) = x$ and $g(x) = |x|$.

3. If $f(x)$ & $g(x)$ both are not differentiable at $x = a$ then the product function ; $F(x) = f(x) . g(x)$ can still be differentiable at $x = a$ e.g. $f(x) = |x|$ & $g(x) = |x|$.

4. If $f(x)$ & $g(x)$ both are non-deri. at $x = a$ then the sum function $F(x) = f(x) + g(x)$ may be a differentiable function . e.g. $f(x) = |x|$ & $g(x) = -|x|$.

5. If $f(x)$ is derivable at $x = a \Rightarrow f'(x)$ is continuous at $x = a$.

e.g. $f(x) = \begin{bmatrix} x^2\sin\frac{1}{x} & \text{if } x\neq 0 \\ 0 & \text{if } x=0 \end{bmatrix}$

G. FUNCTIONAL EQUATIONS

Ex.38 Let $f(xy) = xf(y) + yf(x)$ for all $x, y \in R^+$ and $f(x)$ be differentiable in $(0, \infty)$ then determine $f(x)$.
Given $f(xy)= xf(y) + yf(x)$

Sol. Replacing x by 1 and y by x then we get $x f(1) = 0$ $\therefore$ $f(1) = 0, x \neq 0$ $(\because x, y, \in R^+)$

Now, $$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{f\left(x\left(1+\frac{h}{x}\right)\right)-f(x)}{h} = \lim_{h\to 0}\frac{xf\left(1+\frac{h}{x}\right)+\left(1+\frac{h}{x}\right)f(x)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{xf\left(1+\frac{h}{x}\right)+\frac{h}{x}f(x)}{h} = \lim_{h\to 0}\frac{f\left(1+\frac{h}{x}\right)}{\left(\frac{h}{x}\right)} + \lim_{h\to 0}\frac{f(x)}{x} = f'(x) + \frac{f(x)}{x}$$

$$\Rightarrow f'(x) - \frac{f(x)}{x} = f'(1) \Rightarrow \frac{xf'(x)-f(x)}{x^2} = \frac{f'(1)}{x} \Rightarrow \frac{d}{dx}\left\{\frac{f(x)}{x}\right\} = \frac{f'(1)}{x}$$

On integrating w.r.t.x and taking limit 1 to x then $\frac{f(x)}{x} - \frac{f(1)}{1} = f'(1)(\ln x - \ln 1)$

$\Rightarrow \frac{f(x)}{x} - 0 = f'(1) \ln x$ $\quad \because f(1) = 0)$ $\quad \therefore$ $f(x) = f'(1)(x \ln x)$

Alternative Method :
Given $f(xy) = xf(y) + yf(x)$
Differentiating both sides w.r.t.x treating y as constant, $f'(xy) . y = f(y) + yf'(x)$
Putting y = x and x = 1, then

$f'(xy). x = f(x) + xf'(x)$ $\Rightarrow \frac{xf'(x)-f(x)}{x^2} = \frac{f'(1)}{x}$ $\Rightarrow \frac{d}{dx}\left(\frac{f(x)}{x}\right) = \frac{f'(1)}{x}$

Integrating both sides w.r.t.x taking limit 1 to x,

$\frac{f(x)}{x} - \frac{f(1)}{1} = f'(1)\{\ln x - \ln 1\} \Rightarrow \frac{f(x)}{x} - 0 = f'(1) \ln x$ $\quad (\because f(1) = 0)$

Hence, $f(x) =- f'(1)(x \ln x)$.

Ex.39 If $e^{-xy}f(xy) = e^{-x}f(x) + e^{-y}f(y)\ \forall x, y \in R^+$, and $f'(1) = e$, determine $f(x)$.

Sol. Given $e^{-xy} f(xy) = e^{-x}f(x) + e^{-y}f(y)$ ...(1)

Putting $x = y = 1$ in (1), we get $f(1) = 0$ ...(2)

$$\text{Now, } f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{f\left(x\left(1+\frac{h}{x}\right)\right)-f(x.1)}{h}$$

$$= \lim_{h\to 0}\frac{e^{x+h}.\left\{e^{-x}f(x)+e^{-1\frac{h}{x}}f\left(1+\frac{h}{x}\right)\right\}-2^x(e^{-x}f(x)+e^{-1}f(1))}{h} = \lim_{h\to 0}\frac{e^{h}f(x)+e^{x+h-1\frac{h}{x}}f\left(1+\frac{h}{x}\right)-f(x)-e^{x-1}f(1)}{h}$$

$$= f(x)\lim_{h\to 0}\left(\frac{e^h-1}{h}\right) + e^{(x-1)}\lim_{h\to 0}\frac{e^{h-\frac{h}{x}}f\left(1+\frac{h}{x}\right)}{x.\frac{h}{x}} \qquad (\because f(1) = 0)$$

$$= f(x)\,.\,1 + e^{x-1}.\frac{f'(1)}{x} \qquad = f(x) + \frac{e^{x-1}.e}{x} \qquad (\because f'(1) = e)$$

$$f'(x) = f(x) + \frac{e^x}{x} \quad\Rightarrow\quad e^{-x}f'(x) - e^{-x}f(x) = \frac{1}{x} \quad\Rightarrow\quad \frac{d}{dx}(e^{-x}f(x)) = \frac{1}{x}$$

On integrating we have $e^{-x}f(x) = \ln x + c$ at $x = 1$, $c = 0$ $\quad\therefore\quad f(x) = e^x \ln x$

Ex.40 Find a function continuous and derivable for all x and satisfying the functional relation, $f(x + y)\,.\,f(x - y) = f^2(x)$, where x & y are independent variables & $f(0) \neq 0$.

Sol. Put $y = x$ and $x = 0$ to get $f(x)\,.\,f(-x) = f^2(0)$ ..... (1)

$$\text{Now } f'(x) = \underset{h\to 0}{\text{Limit}}\frac{f(x+h)-f(x)}{h} = \underset{h\to 0}{\text{Limit}}\frac{f(x+h)-\frac{f^2(0)}{f(-x)}}{h} = \frac{1}{f(-x)}\,.\,\underset{h\to 0}{\text{Limit}}\frac{f(x+h).f(-x)-f^2(0)}{h}$$

$$= \frac{1}{f(-x)}\underset{h\to 0}{\text{Limit}}\frac{f\left(\frac{h}{2}+\left(\frac{h}{2}+x\right)\right).f\left(\frac{h}{2}-\left(\frac{h}{2}+x\right)\right)-f^2(0)}{h} = \frac{1}{f(-x)}\underset{h\to 0}{\text{Limit}}\frac{f^2\left(\frac{h}{2}\right)-f^2(0)}{h}$$

$$= \frac{1}{2f(-x)}\underset{h\to 0}{\text{Limit}}\frac{\left[f\left(\frac{h}{2}\right)+f(0)\right]\left[f\left(\frac{h}{2}\right)-f(0)\right]}{\frac{h}{2}} = \frac{f(x)}{2f^2(x)}\,.\,2f(0)\,.\,f'(0) = \frac{f'(0)}{f(0)}f(x)$$

$$\Rightarrow \frac{f'(x)}{f(x)} = \frac{f'(0)}{f(0)} = k \Rightarrow \text{Result}$$

Ex.41 Let f be a function such that $f(x + f(y)) = f(f(x)) + f(y)$ $\forall$ x, y $\in$ R and $f(h) = h$ for $0 < h < \varepsilon$ where $\varepsilon > 0$, then determine $f''(x)$ and $f(x)$.

Sol. Given $f(x + f(y)) = f(f(x) + f(y))$ .....(1)

Put $x = y = 0$ in (1), then $f(0 + f(0)) = f(f(0)) + f(0) \Rightarrow f(f(0)) = f(f(0)) + f(0)$

$\therefore$ $f(0) = 0$ ....(2)

Now $f'(x) = \lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$ (for $0 < h < \varepsilon$)

$= \lim\limits_{h\to 0}\dfrac{f(f(h))}{h}$ {from (1)} $= \lim\limits_{h\to 0}\dfrac{f(h)}{h}$ ($\because$ f(h) = h) $= \lim\limits_{h\to 0}\dfrac{h}{h} = 1.$

Integrating both sides with limites 0 to x then $f(x) = x$ $\therefore$ $f'(x) = 1$.

Ex.42 Let $f : R^+ \to R$ satisfies the functional equation $f(xy) = e^{xy-x-y}\{e^y f(x) + e^x f(y)\}$ $\forall x, y \in R^+$. If $f'(1) = e$, determine $f(x)$.

Sol. Given that; $f(xy) = e^{xy-x-y}\{e^y f(x) + e^x f(y)\}$ $\forall$ x, y $\in R^+$ ...(i)

Putting x = y = 1, we get $f(1) = e^{-1}\{e^1 f(1) + e^1 f(1)\}$ $\Rightarrow$ $f(1) = 0$ ...(ii)

Now, $f'(x) = \lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0}\dfrac{f\left\{x\left(1+\frac{h}{x}\right)\right\}-f(x)}{h}$

$$= \lim_{h\to 0}\frac{e^{x\left(1+\frac{h}{x}\right)-x-\left(1+\frac{h}{x}\right)}\left\{e^{1+\frac{h}{x}}f(x)+e^x f\left(1+\frac{h}{x}\right)\right\}-f(x)}{h} = \lim_{h\to 0}\frac{e^h f(x)+e^{h-1-\frac{h}{x}+x}f\left(1+\frac{h}{x}\right)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x)(e^{h-1})+e^{h-1-\frac{h}{x}+x}\left\{f\left(1+\frac{h}{x}\right)-f(1)\right\}}{h} = \lim_{h\to 0}\frac{f(x)(e^h-1)}{h}+\lim_{h\to 0}\frac{e^{h-1-\frac{h}{x}+x}\left\{f\left(1+\frac{h}{x}\right)-f(1)\right\}}{\frac{h}{x}.x} \quad \{\because f(1) = 0\}$$

$$= f(x) + \frac{e^{x-1}.f'(1)}{x}\left\{\because \lim_{h\to 0}\frac{f\left(1+\frac{h}{x}\right)-f(1)}{\frac{h}{x}} = f'(1) \text{ and } \lim_{h\to 0}\frac{e^h-1}{h} = 1\right\} = f(x) + \frac{e^x}{ex}\,.\,f'(1) \quad \{\because f'(1) = e\}$$

$\therefore f'(x) = f(x) + \dfrac{e^x}{x}$ $\Rightarrow$ $\dfrac{e^x}{x} = f'(x) - f(x)$

$\Rightarrow$ $\dfrac{1}{x} = \dfrac{e^x f'(x) - f(x).e^x}{e^{2x}}$ $\left\{\text{As by quotient rule we can write } \dfrac{e^x f'(x)-f(x).e^x}{(e^x)^2} = \dfrac{d}{dx}\left\{\dfrac{f(x)}{e^x}\right\}\right\}$

$\therefore$ $\dfrac{1}{x} = \dfrac{d}{dx}\left\{\dfrac{f(x)}{e^x}\right\}$

Integrating both sides w.r.t. 'x', we get, $\log|x| + c = \dfrac{f(x)}{e^x}$ or $f(x) = e^x\{\log|x| + c\}$

Since $f(1) = 0 \Rightarrow c = 0$ Thus $f(x) = e^x \log|x|$.

EXERCISE – I SINGLE CORRECT (OBJECTIVE QUESTIONS)

1. A function f(x) is defined as below

$f(x) = \dfrac{\cos(\sin x) - \cos x}{x^2}$, $x \neq 0$ and $f(0) = a$, f(x) is continuous at x = 0 if a equals

(A) 0 (B) 4 (C) 5 (D) 6

Sol.

2. $f(x) = \begin{cases} \dfrac{\sqrt{(1+px)} - \sqrt{(1-px)}}{x} & ,-1 \leq x < 0 \\ \dfrac{2x+1}{x-2} & ,0 \leq x \leq 1 \end{cases}$ is

continuous in the interval [–1, 1], then 'p' is equal to:

(A) –1 (B) – 1/2 (C) 1/2 (D) 1

Sol.

3. Let $f(x) = \left|\left(x + \dfrac{1}{2}\right)[x]\right|$ when $-2 \leq x \leq 2$. Then

(where [∗] represents greatest integer function)

(A) f(x) is continuous at x = 2

(B) f(x) is continuous at x = 1

(C) f(x) is continuous at x = –1

(D) f(x) is discontinuous at x = 0

Sol.

4. Let f(x) = sgn (x) and $g(x) = x(x^2 - 5x + 6)$. The function f(g(x)) is discontinuous at

(A) infinitely many points (B) exactly one point

(C) exactly three points (D) no point

Sol.

5. If $y = \dfrac{1}{t^2 + t - 2}$ where $t = \dfrac{1}{x-1}$, then the number of points of discontinuities of y = f(x), $x \in R$ is

(A) 1 (B) 2 (C) 3 (D) infinite

Sol.

6. The equation 2 tan x + 5x – 2 = 0 has

(A) no solution

(B) at least one real solution in [0, π/4]

(C) two real solution in [0, π/4]

(D) None of these

Sol.

7. If $f(x) = x(\sqrt{x} - \sqrt{x+1})$, then indicate the correct alternative(s)

(A) f(x) is continuous but not differentiable at x = 0

(B) f(x) is differentiable at x = 0

(C) f(x) is not differentiable at x = 0

(D) None of these

Sol.

8. If $f(x) = \begin{cases} \dfrac{x(3e^{1/x}+4)}{2-e^{1/x}} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$ then f(x) is

(A) continuous as well differentiable at x = 0
(B) continuous but not differentiable at x = 0
(C) neither differentiable at x = 0 not continuous at x = 0
(D) none of these

Sol.

9. If $f(x) = \dfrac{x}{\sqrt{x+1}-\sqrt{x}}$ be a real valued function then

(A) f(x) is continuous, but f′(0) does not exist
(B) f(x) is differentiable at x = 0
(C) f(x) is not continuous at x = 0
(D) f(x) is not differentiable at x = 0

Sol.

10. The function $f(x) = \sin^{-1}(\cos x)$ is

(A) discontinuous at x = 0 (B) continuous at x = 0
(C) differentiable at x = 0 (D) none of these

Sol.

11. If $f(x) = \begin{cases} \dfrac{x^2-1}{x^2+1} & , 0 < x \le 2 \\ \dfrac{1}{4}(x^3 - x^2) & , 2 < x \le 3 \\ \dfrac{9}{4}(|x-4| + |2-x|) & , 3 < x < 4 \end{cases}$, then

(A) f(x) is differentiable at x = 2 & x = 3
(B) f (x) is non–differentiable at x = 2 & x = 3
(C) f(x) is differentiable at x = 3 but not at x = 2
(D) f(x) is differentiable at x = 2 but not at x = 3.

Sol.

12. Let f(x) be defined in [–2, 2] by

$f(x) = \begin{cases} \max(\sqrt{4-x^2}, \sqrt{1+x^2}) & , -2 \le x \le 0 \\ \min(\sqrt{4-x^2}, \sqrt{1+x^2}) & , 0 < x \le 2 \end{cases}$ then f(x)

(A) is continuous at all points
(B) is not continuous at more than one point
(C) is not differentiable only at one point
(D) is not differentiable at more than one point.

Sol.

13. The number of points at which the function $f(x) = \max.\{a - x, a + x, b\}$ $-\infty < x < \infty$, $0 < a < b$ cannot be differentiable is

(A) 1 (B) 2
(C) 3 (D) none of these

Sol.

14. If f(x) is differentiable everywhere, then

(A) | f | is differentiable everywhere
(B) $|f|^2$ is differentiable everywhere
(C) f | f | is not differentiable at some point
(D) f + | f | is differentiable everywhere

Sol.

15. Let f(x + y) = f(x) f(y) all x and y. Suppose that f(3) = 3 and f′(0) = 11 then f′(3) is given by
(A) 22 (B) 44 (C) 28 (D) 33
Sol.

16. If $f : R \to R$ be a differentiable function, such that $f(x + 2y) = f(x) + f(2y) + 4xy\ \forall\ x, y \in R$, then
(A) $f'(1) = f'(0) + 1$ (B) $f'(1) = f'(0) - 1$
(C) $f'(0) = f'(1) + 2$ (D) $f'(0) = f'(1) - 2$
Sol.

17. Let $f(x) = x - x^2$ and $g(x) = \begin{cases} \max f(t), 0 \le t \le x, 0 \le x \le 1 \\ \sin \pi x, \quad x > 1 \end{cases}$.
Then in the interval $[0, \infty)$
(A) g(x) is everywhere continuous except at two points
(B) g(x) is everywhere differentiable except at two points
(C) g(x) is everywhere differentiable except at x = 1
(D) none of these
Sol.

18. Let [x] denote the integral part of $x \in R$ and $g(x) = x - [x]$. Let f(x) be any continuous function with f(0) = f(1) then the function h(x) = f(g(x))
(A) has finitely many discontinuities
(B) is continuous on R
(C) is discontinuous at some x = c
(D) is a constant function.

Sol.

19. The function f(x) is defined by
$$f(x) = \begin{cases} \log_{(4x-3)}(x^2 - 2x + 5) & \text{if } \frac{3}{4} < x < 1 \ \& \ x > 1 \\ 4 & \text{if } x = 1 \end{cases}$$
(A) is continuous at x = 1
(B) is discontinuous at x = 1 since $f(1^+)$ does not exist though $f(1^-)$ exists
(C) is discontinuous at x = 1 since $f(1^-)$ does not exist though $f(1^+)$ exists
(D) is discontinuous since neither $f(1^-)$ nor $f(1^+)$ exists.
Sol.

20. Let $f(x) = \begin{cases} x^2 & \text{if } x \text{ is irrational} \\ 1 & \text{if } x \text{ is rational} \end{cases}$ then
(A) f(x) is discontinuous for all x
(B) discontinuous for all x except at x = 0
(C) discontinuous for all x except at x = 1 or – 1
(D) none of these
Sol.

21. A point where function f(x) is not continuous where f(x) = [sin [x]] in (0, 2π) ; is
([∗] denotes greatest integer ≤ x)
(A) (3, 0) (B) (2, 0) (C) (1, 0) (D) (4, – 1)
Sol.

22. The function f defined by $f(x)=\lim_{t\to\infty}\left\{\frac{(1+\sin\pi x)^t-1}{(1+\sin\pi x)^t+1}\right\}$ is

(A) everywhere continuous

(B) discontinuous at all integer values of x

(C) continuous at x = 0 (D) none of these

Sol.

23. If $f(x)=\begin{cases}\sqrt{x}\left(1+\sin\frac{1}{x}\right) & , \ x>0\\ -\sqrt{-x}\left(1+\sin\frac{1}{x}\right) & , \ x<0\\ 0 & , \ x=0\end{cases}$, then f(x) is

(A) continuous as well diff. at x = 0

(B) continuous at x = 0, but not diff. at = 0

(C) neither continuous at x=0 nor diff. at x=0

(D) none of these

Sol.

24. The functions defined by f(x) = max $\{x^2, (x-1)^2, 2x(1-x)\}$, $0 \le x \le 1$

(A) is differentiable for all x

(B) is differentiable for all x except at one point

(C) is differentiable for all x except at two points

(D) is not differentiable at more than two points

Sol.

25. Let f(x) = [n + p sin x], $x \in (0, \pi)$, $n \in I$ and p is a prime number. Then number of points where f(x) is not differentiable is

(where [*] denotes greatest integer function)

(A) p – 1 (B) p + 1 (C) 2p + 1 (D) 2p – 1

Sol.

26. Let $f(x) = x^3 - x^2 + x + 1$ and

$g(x)=\begin{cases}\max\{f(t)\} \text{ for } 0\le t\le x & \text{for } 0\le x\le 1\\ 3-x+x^2 & \text{for } 1<x\le 2\end{cases}$ then

(A) g(x) is continuous & derivable at x = 1

(B) g(x) is continuous but not derivable at x = 1

(C) g(x) is neither continuous nor derivable at x = 1

(D) g(x) is derivable but not continuous at x = 1

Sol.

27. Let f″(x) be continuous at x = 0 and f″(0) = 4 then value of $\lim_{x\to 0}\frac{2f(x)-3f(2x)+f(4x)}{x^2}$ is

(A) 11 (B) 2 (C) 12 (D) none of these

Sol.

28. Let f : R → R be a function such that

$f\left(\frac{x+y}{3}\right)=\frac{f(x)+f(y)}{3}$, f(0) = 0 and f′(0) = 3, then

(A) $\frac{f(x)}{x}$ is differentiable in R

(B) f(x) is continuous but not differentiable in R

(C) f(x) is continuous in R (D) f(x) is bounded in R

Sol.

29. Suppose that f is a differentiable function with the property that $f(x + y) = f(x) + f(y) + xy$ and $\lim\limits_{h\to 0}\frac{1}{h}$ $f(h) = 3$ then

(A) f is a linear function (B) $f(x) = 3x + x^2$

(C) $f(x) = 3x + \frac{x^2}{2}$ (D) none of these

Sol.

30. If a differentiable function f satisfies

$f\left(\frac{x+y}{3}\right) = \frac{4-2(f(x)+f(y))}{3}$ $\forall$ x, y $\in$ R, find f(x)

(A) 1/7 (B) 2/7 (C) 8/7 (D) 4/7

Sol.

31. Let f : R $\to$ R be a function defined by f(x) = Min $\{x + 1, |x| + 1\}$. Then which of the following is true ?

(A) $f(x) \geq 1$ for all $x \in R$

(B) f(x) is not differentiable at x = 1

(C) f(x) is differentiable everywhere

(D) f(x) is not differentiable at x = 0

Sol.

32. The function f : R /{0} $\to$ R given by

$f(x) = \frac{1}{x} - \frac{2}{e^{2x}-1}$ can be made continuous at x = 0 by defining f(0) as

(A) 2 (B) –1 (C) 0 (D) 1

Sol.

33. If f is a real-valued differentiable function satisfying $|f(x) - f(y)| \leq (x - y)^2$, x, y $\in$ R and f(0) = 0, then f(1) equals

(A) 1 (B) 2 (C) 0 (D) –1

Sol.

34. Function $f(x) = (|x - 1| + |x - 2| + \cos x)$ where $x \in [0, 4]$ is not continuous at number of points

(A) 3 (B) 2 (C) 1 (D) 0

Sol.

35. If $f(x) = \begin{cases} \frac{1-|x|}{1+x} & , x \neq -1 \\ 1 & , x = -1 \end{cases}$, then f([2x]) is

(where [$*$] represent greatest integer function)

(A) continuous at x = –1 (B) continuous at x=0

(C) discontinuous at x = 1/2 (D) all to these

Sol.

36. Let $f(x + y) = f(x)\,f(y)$ for all x, y, where $f(0) \neq 0$. If $f'(0) = 2$, then f(x) is equal to

(A) Ae^x (B) e^{2x} (C) 2x (D) None of these

Sol.

37. A function $f : R \to R$ satisfies the equation $f(x + y) = f(x) \,.\, f(y)$ for all $x, y \in R$, $f(x) \neq 0$. Suppose that the function is differentiable at x = 0 and $f'(0) = 2$ then $f'(x) =$

(A) f(x) (B) 2 f(x) (C) – f(x) (D) – 2 f(x)

Sol.

38. The value of f(0), so that the function,

$f(x)=\dfrac{\sqrt{(a^2-ax+x^2)}-\sqrt{(a^2+ax+x^2)}}{\sqrt{(a+x)}-\sqrt{(a-x)}}$ (a > 0) becomes continuous for all x, is given by

(A) $a\sqrt{a}$ (B) $\sqrt{a}$ (C) $-\sqrt{a}$ (D) $-a\sqrt{a}$

Sol.

39. If $f(x) = \begin{cases} \dfrac{\sin\{\cos x\}}{x-\frac{\pi}{2}} & , \ x \neq \dfrac{\pi}{2} \\ 1 & , \ x = \dfrac{\pi}{2} \end{cases}$, then f(x) is

(where { * } represents the fractional part function)

(A) continuous at $x = \dfrac{\pi}{2}$

(B) $\lim\limits_{x\to\frac{\pi}{2}} f(x)$ exists, but f is not continuous at $x = \dfrac{\pi}{2}$

(C) $\lim\limits_{x\to\frac{\pi}{2}} f(x)$ does not exist (D) $\lim\limits_{x\to\frac{\pi}{2}} f(x) = 1$

Sol.

40. Let $f(x) = [\cos x + \sin x]$, $0 < x < 2\pi$ where [x] denotes the greatest integer less than or equal to x. the number of points of discontinuity of f(x) is

(A) 6 (B) 5 (C) 4 (D) 3

Sol.

41. The function $f(x) = \begin{cases} x^2\left[\dfrac{1}{x^2}\right] & ; \ x \neq 0 \\ 0 & ; \ x = 0 \end{cases}$, is [x] represents the greatest integer less than or equal to x

(A) continuous at x = 1 (B) continuous at x = –1

(C) continuous at x = 0 (D) continuous at x = 2

Sol.

42. The function $f(x) = \begin{bmatrix} \sin(\ln|x|) & x \neq 0 \\ 1 & x = 0 \end{bmatrix}$

(A) is continuous at x = 0

(B) has removable discontinuity at x = 0

(C) has jump discontinuity at x = 0

(D) has discontinuity of II^{nd} type at x = 0

Sol.

43. The set of all point for which $f(x) = \frac{|x-3|}{|x-2|} + \frac{1}{[1+x]}$ is continuous is
(where [$*$] represents greatest integer function)
(A) R (B) $R - [-1, 0]$
(C) $R - (\{2\} \cup [-1, 0])$ (D) $R - \{(-1, 0) \cup n, n \in I\}$
Sol.

44. Let f(x) be a continuous function defined for $1 \le x \le 3$. If f(x) takes rational values of for all x and f(2) = 10 then the value of f(1.5) is
(A) 7.5 (B) 10 (C) 8 (D) None of these
Sol.

45. If [x] and {x} represents integral and fractional parts of a real number x, and $f(x) = \frac{a^{2[x]+\{x\}} - 1}{2[x] + \{x\}}$, $x \neq 0$, $f(0) = \log_e a$, where $a > 0$, $a \neq 1$, then
(A) f(x) is continuous at x = 0
(B) f(x) has a removable discontinuity at x = 0
(C) $\lim_{x \to 0} f(x)$ does not exist (D) None of these
Sol.

46. If $f(x) = p\,|\sin x| + q \,.\, e^{|x|} + r|x|^3$ and f(x) is differentiable at x = 0, then
(A) p = q = r = 0 (B) p = 0, q = 0, $r \in R$
(C) q = 0, r = 0, $p \in R$ (D) p + q = 0, $r \in R$
Sol.

47. Let f(x) = sin x, g(x) = [x + 1] and g(f(x)) = h(x) then $h'\left(\frac{\pi}{2}\right)$ is (where [$*$] is the greatest integer function)
(A) nonexistent (B) 1
(C) –1 (D) None of these
Sol.

48. If $f(x) = [\tan^2 x]$ then
(where [$*$] denotes the greatest integer function)
(A) $\lim_{x \to 0} f(x)$ does not exist
(B) f(x) is continuous at x = 0
(C) f(x) is non-differentiable at x = 0 (D) f(0) = 1
Sol.

49. If $f(x) = [x]^2 + \sqrt{\{x\}^2}$, then
(where, [$*$] and { $*$ } denote the greatest integer and fractional part functions respectively)
(A) f(x) is continuous at all integral points
(B) f(x) is continuous and differentiable at x = 0
(C) f(x) is discontinuous $\forall$ $x \in I - \{1\}$
(D) f(x) is differentiable $\forall$ $x \in I$.
Sol.

50. If f is an even function such that $\lim\limits_{h\to 0^+}\dfrac{f(h)-f(0)}{h}$ has some finite non-zero value, then
(A) f is continuous and derivable at x = 0
(B) f is continuous but not derivable at x = 0
(C) f may be discontinuous at x = 0
(D) None of these
Sol.

51. f is a continuous function on the real line. Given that $x^2 + (f(x) - 2)x - \sqrt{3} \cdot f(x) + 2\sqrt{3} - 3 = 0$. then the value of $f(\sqrt{3})$
(A) can not be determined (B) is $2(1 - \sqrt{3})$
(C) is zero (D) is $\dfrac{2(\sqrt{3}-2)}{\sqrt{3}}$
Sol.

52. If f(x) = sgn (cos 2x – 2 sin x + 3) then f(x) (where sgn ( ) is the signum function)
(A) is continuous over its domain
(B) has a missing point discontinuity
(C) has isolated point discontinuity
(D) has irremovable discontinuity.
Sol.

53. Let $g(x) = \tan^{-1}|x| - \cot^{-1}|x|$, $f(x) = \dfrac{[x]}{[x+1]}\{x\}$, $h(x) = |g(f(x))|$ then which of the following holds good ?
(where { ∗ } denotes fractional part and [∗] denotes the integral part)
(A) h is continuous at x = 0
(B) h is discontinuous at x = 0
(C) $h(0^-) = \pi/2$ (D) $h(0^+) = -\pi/2$

Sol.

54. Consider $f(x) = \lim\limits_{n\to\infty}\dfrac{x^n - \sin x^n}{x^n + \sin x^n}$ for $x > 0$, $x \neq 1$, f(1) = 0 then
(A) f is continuous at x = 1
(B) f has a finite discontinuity at x = 1
(C) f has an infinite or oscillatory discontinuity at x = 1.
(D) f has a removable type of discontinuity at x = 1.
Sol.

55. Given $f(x) = \begin{cases} \dfrac{[\{|x|\}]e^{x^2}\{[x+\{x\}]\}}{(e^{1/x^2}-1)\,\mathrm{sgn}(\sin x)} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$ then, f(x)
(where {x} is the fractional part function; [x] is the step up function and sgn(x) is the signum function of x)
(A) is continuous at x = 0
(B) is discontinuous at x = 0
(C) has a removable discontinuity at x = 0
(D) has an irremovable discontinuity at x = 0
Sol.

56. Consider $f(x) = \begin{cases} x[x]^2 \log_{(1+x)} 2 & \text{for } -1 < x < 0 \\ \dfrac{\ln(e^{x^2} + 2\sqrt{\{x\}})}{\tan\sqrt{x}} & \text{for } 0 < x < 1 \end{cases}$ the

(where [∗] & { ∗ } are the greatest integer function & fractional part function respectively)

(A) $f(0) = \ln 2 \Rightarrow f$ is continuous at $x = 0$

(B) $f(0) = 2 \Rightarrow f$ is continuous at $x = 0$

(C) $f(0) = e^2 \Rightarrow f$ is continuous at $x = 0$

(D) f has an irremovable discontinuity at $x = 0$

Sol.

57. Consider $f(x) = \dfrac{\sqrt{1+x} - \sqrt{1-x}}{\{x\}}$, $x \neq 0$; $g(x) = \cos 2x$, $-\dfrac{\pi}{4} < x < 0$, $h(x) = \begin{cases} \dfrac{1}{\sqrt{2}} f(g(x)) & \text{for } x < 0 \\ 1 & \text{for } x = 0 \\ f(x) & \text{for } x > 0 \end{cases}$

then, which of the following holds good

(where { ∗ } denotes fractional part function)

(A) 'h' is continuous at $x = 0$

(B) 'h' is discontinuous at $x = 0$

(C) $f(g(x))$ is an even function

(D) $f(x)$ is an even function

Sol.

58. Consider the function defined on $[0, 1] \to R$,

$f(x) = \dfrac{\sin x - x\cos x}{x^2}$ if $x \neq 0$ and $f(0) = 0$, then the function $f(x)$

(A) has a removable discontinuity at $x = 0$

(B) has a non removable finite discontinuity at $x = 0$

(C) has a non removable infinite discontinuity at $x = 0$

(D) is continuous at $x = 0$

Sol.

59. Let $f(x) = \dfrac{|x|}{\sin x}$ for $x \neq 0$ & $f(0) = 1$ then ,

(A) $f(x)$ is conti. & diff. at $x = 0$

(B) $f(x)$ is continuous & not derivable at $x = 0$

(C) $f(x)$ is discont. & not diff. at $x = 0$

(D) None of these

Sol.

60. Given

$f(x) = \begin{cases} \log_a(a\,|[x]+[-x]|)^x \left(\dfrac{a^{\left(\frac{[x]+[-x]}{|x|}\right)^{\frac{2}{\,}} - 5}}{3 + a^{\frac{1}{|x|}}} \right) & \text{for } |x| \neq 0 ; a > 1 \\ 0 & \text{for } x = 0 \end{cases}$ then

(where [∗] represent the integral part function)

(A) f is continuous but not differentiable at $x = 0$

(B) f is cont. & diff. at $x = 0$

(C) the differentiability of 'f' at $x = 0$ depends on the value of a

(D) f is cont. & diff. at $x = 0$ and for $a = e$ only.

Sol.

61. For what triplets of real number (a, b, c) with $a \neq 0$ the function $f(x) = \begin{bmatrix} x & x \le 1 \\ ax^2 + bx + c & \text{otherwise} \end{bmatrix}$ is differentiable for all real x ?
(A) $\{(a, 1-2a, a) \mid a \in R, a \neq 0\}$
(B) $\{(a, 1-2a, c) \mid a, c \in R, a \neq 0\}$
(C) $\{(a, b) \mid a, b, c \in R, a + b + c = 1\}$
(D) $\{(a, 1-2a, 0) \mid a \in R, a \neq 0\}$
Sol.

62. A function f defined as $f(x) = x[x]$ for $-1 \le x \le 3$ where [x] defines the greatest integer $\le x$ is
(A) conti. at all points in the domain of but non-derivable at a finite number of points
(B) discontinuous at all points & hence non-derivable at all points in the domain of f
(C) discont. at a finite number of points but not derivable at all points i the domain of f
(D) discont. & also non-derivable at a finite number of points of f.
Sol.

63. The function f(x) is defined as follows
$f(x) = \begin{bmatrix} -x & \text{if} & x < 0 \\ x^2 & \text{if} & 0 \le x \le 1 \\ x^3 - x + 1 & \text{if} & x > 1 \end{bmatrix}$ then f(x) is
(A) derivable & cont. at x = 0
(B) derivable at x = 1 but not cont. at x = 1
(C) neither derivable nor cont. at x = 1
(D) not derivable at x = 0 but cont. at x = 1
Sol.

64. If $f(x) = \begin{bmatrix} x + \{x\} + x\sin\{x\} & \text{for} & x \neq 0 \\ 0 & \text{for} & x = 0 \end{bmatrix}$ then
(where { ∗ } denotes the fractional part function)
(A) 'f' is cont. & diff. at x = 0
(B) 'f' is cont. but not diff. at x = 0
(C) 'f' is cont. & diff. at x = 2 (D) None of these
Sol.

EXERCISE – II MULTIPLE CORRECT (OBJECTIVE QUESTIONS)

1. Which of the following function(x) not defined at x = 0 has/have non-removable discontinuity at the origin ?

(A) $f(x) = \dfrac{1}{1+2^{\frac{1}{x}}}$ (B) $f(x) = \tan^{-1}\dfrac{1}{x}$

(C) $f(x) = \dfrac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}$ (D) $f(x) = \dfrac{1}{\ln|x|}$

Sol.

2. Which of the following function(s) defined below has / have single point continuity.

(A) $f(x) = \begin{bmatrix} 1 & \text{if } x \in Q \\ 0 & \text{if } x \notin Q \end{bmatrix}$ (B) $g(x) = \begin{bmatrix} x & \text{if } x \in Q \\ 1-x & \text{if } x \notin Q \end{bmatrix}$

(C) $h(x) = \begin{bmatrix} x & \text{if } x \in Q \\ 0 & \text{if } x \notin Q \end{bmatrix}$ (D) $k(x) = \begin{bmatrix} x & \text{if } x \in Q \\ -x & \text{if } x \notin Q \end{bmatrix}$

Sol.

3. The function $f(x) = \begin{bmatrix} |x-3| & ,x \ge 1 \\ \left(\dfrac{x^2}{4}\right)-\left(\dfrac{3x}{2}\right)+\left(\dfrac{13}{4}\right) & ,x < 1 \end{bmatrix}$ is

(A) continuous at x = 1 (B) differentiable at x = 1
(C) continuous at x = 3 (D) differentiable at x = 3

Sol.

4. The points at which the function,
$f(x) = |x - 0.5| + |x - 1| + \tan x$ does not have a derivative in the interval (0, 2) are
(A) 1 (B) $\pi/2$ (C) 3 (D) 1/2

Sol.

5. Let f(x) and g(x) be defined by f(x) = [x] and
$g(x) = \begin{cases} 0 , & x \in I \\ x^2 , & x \in R-I \end{cases}$ then
(where [*] denotes the greatest integer function)

(A) $\lim\limits_{x\to1} g(x)$ exists, but g is not continuous at x = 1

(B) $\lim\limits_{x\to1} f(x)$ does not exist and f is not continuous at x = 1.

(C) gof is continuous for all x

(D) fog is continuous for all x

Sol.

6. Given f(x) = $\begin{cases} \left[3-\left[\cot^{-1}\left(\dfrac{2x^3-3}{x^2}\right)\right]\right] & \text{for } x>0 \\ \{x^2\}\cos(e^{1/x}) & \text{for } x<0 \end{cases}$ then

which of the following statement does not hold good. (where { * } & [*] denotes the fractional part and the integral part function respectively)
(A) $f(0^-) = 0$
(B) $f(0^+) = 3$
(C) $f(0) = 0 \Rightarrow$ continuity of f at x = 0
(D) irremovable discontinuity of f at x = 0
Sol.

7. Let $f(x) = [x] + \sqrt{x-[x]}$. Then
(where [*] denotes the greatest integer function)
(A) f(x) is continuous on R^+
(B) f(x) is continuous on R
(C) f(x) is continuous on R – I
(D) discontinuous at x = 1
Sol.

8. If $f(x) = \sum_{k=0}^{n} a_k |x|^k$, where a_i's are real constants, then f(x) is
(A) continuous at x = 0 for all a_i
(B) differentiable at x = 0 for all $a_i \in R$
(C) differentiable at x = 0 for all $a_{2k+1} = 0$
(D) none of these
Sol.

9. Let [x] denotes the greatest integer less than or equal to x. If f(x) = [x sin πx], then f(x) is
(A) continuous at x = 0
(B) continuous in (–1, 0)
(C) differentiable at x=1
(D) differentiable in (–1, 1)
Sol.

10. Let $f(x) = \lim_{n\to\infty} (\sin x)^{2n}$, then f is
(A) continuous at x = π/2
(B) discontinuous at x = π/2
(C) discontinuous at x = 0
(D) discontinuous at an infinite number of points
Sol.

11. Let $f(x) = \dfrac{1}{[\sin x]}$ then
(where [*] denotes the greatest integer function)
(A) domain of f(x) is $(2n\pi + \pi, 2n\pi + 2\pi) \cup \{2n\pi + \pi/2\}$
(B) f(x) is continuous when $x \in (2n\pi + \pi, 2n\pi + 2\pi)$
(C) f(x) is continuous at $x = 2n\pi + \pi/2$
(D) f(x) has the period 2π

Sol.

12. The function $f(x) = \sqrt{1-\sqrt{1-x^2}}$

(A) has its domain $-1 \le x \le 1$
(B) both $f'(0^-)$ and $f'(0)$ are finite
(C) is continuous and differentiable at $x = 0$
(D) is continuous but not differentiable at $x = 0$

Sol.

13. Let $f(x) = \lim\limits_{n\to\infty} \dfrac{1-x^n}{1+x^n}$. Then

(A) f(x) is a constant in $0 < x < 1$
(B) f(x) is continuous at $x = 1$
(C) f(x) is not differentiable at $x = 1$
(D) None of these

Sol.

14. Let 'f' be a continuous function on R.

If $f(1/4^n) = (\sin e^n)\ e^{-n^2} + \dfrac{n^2}{n^2+1}$ then f(0) is

(A) not unique (B) 1
(C) data sufficient to find f(0)
(D) data insufficient of find f(0)

Sol.

15. Indicate all correct alternatives if, $f(x) = \dfrac{x}{2} - 1$, then on the interval $[0, \pi]$

(A) $\tan(f(x))$ & $\dfrac{1}{f(x)}$ are both continuous
(B) $\tan(f(x))$ & $\dfrac{1}{f(x)}$ are both discontinuous
(C) $\tan(f(x))$ & $f^{-1}(x)$ are both continuous
(D) $\tan(f(x))$ is continuous but $\dfrac{1}{f(x)}$ is not

Sol.

16. $f(x) = |[x]\,x|$ in $-1 \le x \le 2$, then f(x) is
(where [$*$] denotes greatest integer $\le x$)

(A) cont. at $x = 0$ (B) discont. $x = 0$
(C) not diff. at $x = 2$ (D) diff. at $x = 2$

Sol.

17. $f(x) = 1 + x\,.\,[\cos x]$ in $0 < x \le \pi/2$, then f(x) is
(where [$*$] denotes greatest integer $\le x$)

(A) It is continuous in $0 < x < \pi/2$
(B) It is differentiable in $0 < x < \pi/2$
(C) Its maximum value is 2
(D) Its not differentiable in $0 < x < \pi/2$

Sol.

18. $f(x) = (\sin^{-1} x)^2 . \cos (1/x)$ if $x \neq 0$; $f(0) = 0$, $f(x)$ is
(A) cont. no where in $-1 \le x \le 1$
(B) cont. every where in $-1 \le x \le 1$
(C) differentiable no where in $-1 \le x \le 1$
(D) differentiable everywhere in $-1 < x < 1$
Sol.

19. $f(x) = |x| + |\sin x|$ in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. It is
(A) conti. no where
(B) conti. every where
(C) differentiable no where
(D) Differentiable every where except at $x = 0$
Sol.

20. If $f(x) = 3(2x + 3)^{2/3} + 2x + 3$ then
(A) $f(x)$ is cont. but not diff. at $x = -3/2$
(B) $f(x)$ is diff. at $x = 0$
(C) $f(x)$ is cont. at $x = 0$
(D) $f(x)$ is diff. but not cont. at $x = -3/2$
Sol.

21. If $f(x) = 2 + |\sin^{-1} x|$, it is
(A) continuous no where
(B) continuous every where in its domain
(C) differentiable no where in its domain
(D) Not differentiable at $x = 0$
Sol.

22. If $f(x) = x^2 . \sin (1/x)$, $x \neq 0$ and $f(0) = 0$ then,
(A) $f(x)$ is continuous at $x = 0$
(B) $f(x)$ is derivable at $x = 0$
(C) $f'(x)$ is continuous at $x = 0$
(D) $f'(x)$ is not derivable at $x = 0$
Sol.

23. A function which is continuous & not differentiable at $x = 0$ is
(A) $f(x) = x$ for $x < 0$ & $f(x) = x^2$ for $x \ge 0$
(B) $g(x) = x$ for $x < 0$ & $g(x) = 2x$ for $x \ge 0$
(C) $h(x) = x\,|x|$ $x \in R$ (D) $K(x) = 1 + |x|$, $x \in R$
Sol.

24. If $\sin^{-1} x + |y| = 2y$ then y as a function of x is
(A) defined for $-1 \le x \le 1$
(B) continuous at $x = 0$
(C) differentiable for all x
(D) such that $\frac{dy}{dx} = \frac{1}{3\sqrt{1-x^2}}$ for $-1 < x < 0$
Sol.

EXERCISE – III SUBJECTIVE QUESTIONS

1. If the function $f(x) = \frac{3x^2+ax+a+3}{x^2+x-2}$ is continuous at $x = -2$. Find $f(-2)$.

Sol.

2. Find all possible values of a and b so that f(x) is continuous for all $x \in R$ if

$$f(x) = \begin{cases} |ax+3| & \text{if } x \le -1 \\ |3x+a| & \text{if } -1 < x \le 0 \\ \frac{b\sin 2x}{x} - 2b & \text{if } 0 < x < \pi \\ \cos^2 x - 3 & \text{if } x \ge \pi \end{cases}$$

Sol.

3. Let $f(x) = \begin{cases} \frac{\ln \cos x}{\sqrt[4]{1+x^2}-1} & \text{if } x > 0 \\ \frac{e^{\sin 4x}-1}{\ln(1+\tan 2x)} & \text{if } x < 0 \end{cases}$ Is it possible to define f(0) to make the function continuous at x = 0. If yes what is the value of f(0), if not then indicate the nature of discontinuity.

Sol.

4. Suppose that $f(x) = x^3 - 3x^2 - 4x + 12$ and

$h(x) = \begin{cases} \frac{f(x)}{x-3} & , x \ne 3 \\ K & , x = 3 \end{cases}$ then

(a) find all zeros of f(x)

(b) find the value of K that makes h continuous at x = 3

(c) using the value of K found in (b), determine whether h is an even function.

Sol.

5. Let $y_n(x) = x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \dots\dots\dots + \frac{x^2}{(1+x^2)^{n-1}}$ and $y(x) = \lim_{n\to\infty} y_n(x)$

Discuss the continuity of $y_n(x)$ ($n \in N$) and y(x) at x = 0

Sol.

6. Draw the graph of the function $f(x) = x - |x - x^2|$, $-1 \le x \le 1$ & discuss the continuity or discontinuity of f in the interval $-1 \le x \le 1$.

Sol.

7. Let $f(x) = \begin{cases} \dfrac{1-\sin \pi x}{1+\cos 2\pi x}, & x < \dfrac{1}{2} \\ p, & x = \dfrac{1}{2} \\ \dfrac{\sqrt{2x-1}}{\sqrt{4+\sqrt{2x-1}}-2}, & x > \dfrac{1}{2} \end{cases}$. Determine the value of p, if possible, so that the function is continuous at x = 1/2

Sol.

8. Given the function $g(x) = \sqrt{6-2x}$ and $h(x) = 2x^2 - 3x + a$. Then

(a) evaluate h (g(2))

Sol.

(b) If $f(x) = \begin{cases} g(x), & x \le 1 \\ h(x), & x > 1 \end{cases}$, find 'a' so that f is continuous.

Sol.

9. Let $f(x) = \begin{cases} 1+x \ , & 0 \le x \le 2 \\ 3-x \ , & 2 < x \le 3 \end{cases}$. Determine the form of g(x) = f[f(x)] & hence find the point of discontinuity of g, if any.

Sol.

10. Let [x] denote the greatest integer function & f(x) be defined in a neighbourhood of 2 by

$$f(x) = \begin{cases} \dfrac{(\exp\{(x+2)\ell n 4\})^{\frac{[x+1]}{4}} - 16}{4^x - 16}, & x < 2 \\ A\dfrac{1-\cos(x-2)}{(x-2)\tan(x-2)}, & x > 2 \end{cases}$$

Find the values of A & f(2) in order that f(x) may be continuous at x = 2.

Sol.

11. The function $f(x) = \begin{cases} \left(\dfrac{6}{5}\right)^{\frac{\tan 6x}{\tan 5x}} & \text{if } 0 < x < \dfrac{\pi}{2} \\ b+2 & \text{if } x = \dfrac{\pi}{2} \\ (1+|\cos x|)^{\left(\frac{a|\tan x|}{b}\right)} & \text{if } \dfrac{\pi}{2} < x < \pi \end{cases}$

Determine the values of 'a' & 'b', if f is continuous at x = π/2.

Sol.

12. Determine a & b so that f is continuous at x = $\frac{\pi}{2}$

where $f(x) = \begin{cases} \dfrac{1-\sin^3 x}{3\cos^2 x} & \text{if } x < \dfrac{\pi}{2} \\ a & \text{if } x = \dfrac{\pi}{2} \\ \dfrac{b(1-\sin x)}{(\pi-2x)^2} & \text{if } x > \dfrac{\pi}{2} \end{cases}$

Sol.

13. Determine the values of a, b & c for which the

function $f(x) = \begin{cases} \dfrac{\sin(a+1)x+\sin x}{x} & \text{for } x<0 \\ c & \text{for } x=0 \\ \dfrac{(x+bx^2)^{1/2}-x^{1/2}}{bx^{3/2}} & \text{for } x>0 \end{cases}$

is continuous at x = 0.

Sol.

14. If $f(x) = \dfrac{\sin 3x + A\sin 2x + B\sin x}{x^5}$ $(x \neq 0)$ is cont. at x = 0. Find A & B. Also find f(0).

Sol.

15. Let $f(x) = \begin{cases} \dfrac{\left(\dfrac{\pi}{2}-\sin^{-1}\left(1-\{x\}^2\right)\right)\sin^{-1}(1-\{x\})}{\sqrt{2}\left(\{x\}-\{x\}^3\right)} & \text{for } x \neq 0 \\ \dfrac{\pi}{2} & \text{for } x = 0 \end{cases}$

where {x} is the fractional part of x.

Consider another function g(x); such that

$g(x) = \begin{cases} f(x) & \text{for } x \geq 0 \\ 2\sqrt{2}\, f(x) & \text{for } x < 0 \end{cases}$

Discuss the continuity of the functions f(x) & g(x) at x = 0.

Sol.

16. Discuss the continuity of f in [0, 2] where

$f(x) = \begin{cases} |4x-5|\,[x] & \text{for } x>1 \\ [\cos \pi x] & \text{for } x \leq 1 \end{cases}$; where [x] is the greatest integer not greater than x. Also draw the graph.

Sol.

17. If $f(x) = x + \{-x\} + [x]$, where $[x]$ is the integral part & $\{x\}$ is the fractional part of x. Discuss the continuity of f in $[-2, 2]$.

Sol.

18. Find the locus of (a, b) for which the function

$$f(x) = \begin{cases} ax-b & \text{for} \quad x \le 1 \\ 3x & \text{for} \quad 1<x<2 \\ bx^2-a & \text{for} \quad x \ge 2 \end{cases}$$

is continuous at x = 1 but discontinuous at x = 2.

Sol.

19. A function $f : R \to R$ is defined as

$f(x) = \lim\limits_{n\to\infty} \dfrac{ax^2+bx+c+e^{nx}}{1+c\cdot e^{nx}}$ where f is continuous on R. Find the values of a, b and c.

Sol.

20. Let $g(x) = \lim\limits_{n\to\infty} \dfrac{x^n f(x)+h(x)+1}{2x^n+3x+3}$, $x \neq 1$ and

$g(1) = \lim\limits_{x\to 1} \dfrac{\sin^2(\pi\cdot 2^x)}{\ell n(\sec(\pi\cdot 2^x))}$ be a continuous function at x = 1, find the value of 4 g (1) + 2 f(1) – h(1). Assume that f(x) and h(x) are continuous at x = 1.

Sol.

21. The function $f(x) = \left(\dfrac{2+\cos x}{x^3 \sin x} - \dfrac{3}{x^4}\right)$ is not defined at x = 0. How should the function be defined at x = 0 to make it continuous at x = 0.

Sol.

22. $f(x) = \begin{cases} \dfrac{a^{\sin x}-a^{\tan x}}{\tan x-\sin x} & \text{for } x>0 \\ \dfrac{\ell n(1+x+x^2)+\ell n(1-x+x^2)}{\sec x-\cos x} & \text{for } x<0 \end{cases}$, if f is continuous at x = 0, find 'a'

Now if $g(x) = \ell n\left(2-\dfrac{x}{a}\right)\cot(x-a)$ for $x \neq a$, $a \neq 0$, $a > 0$. If g is continuous at x = a then show that $g(e^{-1}) = -e$.

Sol.

23. Let $f(x + y) = f(x) + f(y)$ for all x, y & if the function f(x) is continuous at x = 0, then show that f(x) is continuous at all x.

Sol.

24. Given $f(x) = \sum_{r=1}^{n} \tan\left(\frac{x}{2^r}\right)\sec\left(\frac{x}{2^{r-1}}\right)$; r, n ∈ N

$$g(x) = \underset{n\to\infty}{\text{Limit}} \frac{\ell n\left(f(x)+\tan\frac{x}{2^n}\right)-\left(f(x)+\tan\frac{x}{2^n}\right)^n \cdot \left[\sin\left(\tan\frac{x}{2}\right)\right]}{1+\left(f(x)+\tan\frac{x}{2^n}\right)^n} = k$$

for $x = \frac{\pi}{4}$ and the domain of g(x) is (0, π/2).

(where [∗] denotes the greatest integer function) Find the value of k, if possible, so that g(x) is continuous at x = π/4. Also state the points of discontinuity of g(x) in (0, π/4), if any.

Sol.

25. Let $f(x) = x^3 - x^2 - 3x - 1$ and $h(x) = \frac{f(x)}{g(x)}$ where h is a rational function such that

(a) it is continuous every where except when x = –1,

(b) $\lim_{x\to\infty} h(x) = \infty$ and **(c)** $\lim_{x\to-1} h(x) = \frac{1}{2}$.

Find $\lim_{x\to 0} (3h(x) + f(x) - 2g(x))$

Sol.

26. (a) If g : [a, b] onto [a, b] is continuous show that there is some c ∈ [a, b] such that g (c) = c.

Sol.

(b) Let f be continuous on the interval [0, 1] to R such that f(0) = f(1). Prove that there exists a point c in $\left[0, \frac{1}{2}\right]$ such that $f(c) = f\left(c+\frac{1}{2}\right)$

Sol.

27. Consider the function

$$g(x) = \begin{cases} \dfrac{1-a^x+xa^x\,\ell na}{a^x\,x^2} & \text{for } x<0 \\[2ex] \dfrac{2^x a^x - x\,\ell n2 - x\,\ell na - 1}{x^2} & \text{for } x>0 \end{cases} \quad \text{where } a > 0.$$

find the value of 'a' & 'g(0)' so that the function g(x) is continuous at x = 0.

Sol.

EXERCISE – IV ADVANCED SUBJECTIVE QUESTIONS

1. Discuss the continuity & differentiability of the function $f(x) = \sin x + \sin |x|$, $x \in R$. Draw a rough sketch of the graph of f(x).
Sol.

2. Examine the continuity and differentiability of $f(x) = |x| + |x - 1| + |x - 2|$ $x \in R$.
Also draw the graph of f(x).
Sol.

3. If the function f(x) defined as

$$f(x) = \begin{cases} -\dfrac{x^2}{2} & \text{for } x \le 0 \\ x^n \sin\dfrac{1}{x} & \text{for } x > 0 \end{cases}$$

is continuous but not derivable at x = 0. Then find the range of n.
Sol.

4. A function f is defined as follows :

$$f(x) = \begin{cases} 1 & \text{for } -\infty < x < 0 \\ 1 + |\sin x| & \text{for } 0 \le x < \dfrac{\pi}{2} \\ 2 + \left(x - \dfrac{\pi}{2}\right)^2 & \text{for } \dfrac{\pi}{2} \le x < +\infty \end{cases}$$

Discuss the continuity & differentiability at x = 0 & $x = \pi/2$.
Sol.

5. Examine the origin for continuity & derivability in the case of the function f defined by
$f(x) = x \tan^{-1}(1/x)$, $x \ne 0$ and $f(0) = 0$.
Sol.

6. Let f(0) = 0 and f '(0) = 1. For a positive integer k, show that

$$\lim_{x \to 0} \frac{1}{x}\left(f(x) + f\left(\frac{x}{2}\right) + \ldots. f\left(\frac{x}{k}\right)\right) = 1 + \frac{1}{2} + \frac{1}{3} + \ldots.. + \frac{1}{k}$$

Sol.

7. Let $f(x) = xe^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)}$; $x \neq 0$, $f(0) = 0$, test the continuity & differentiability at $x = 0$.

Sol.

8. If $f(x) = |x - 1|.\ ([x] - [-x])$, then find $f'(1^+)$ & $f'(1^-)$ (where [*] denotes greatest integer function)

Sol.

9. If $f(x) = \begin{cases} ax^2 - b & \text{if } |x| < 1 \\ -\dfrac{1}{|x|} & \text{if } |x| \geq 1 \end{cases}$ is derivable at $x = 1$. Find the values of a & b.

Sol.

10. Let f(x) be defined in the interval [−2, 2] such that $f(x) = \begin{cases} -1 & , -2 \leq x \leq 0 \\ x-1 & , \ 0 < x \leq 2 \end{cases}$ & $g(x) = f(|x|) + |f(x)|$. Test the differentiability of g(x) in (−2, 2).

Sol.

11. Given $f(x) = \cos^{-1}\left(\text{sgn}\left(\dfrac{2[x]}{3x-[x]}\right)\right)$ Discuss the continuity & differentiability of f(x) at $x = \pm 1$. (where sgn (*) denotes the signum function & [*] denotes the greatest integer function)

Sol.

12. Examine for continuity & differentiabilty the points x = 1 & x = 2, the function f defined by

$$f(x) = \begin{cases} x[x] & , \ 0 \leq x < 2 \\ (x-1)[x] & , \ 2 \leq x \leq 3 \end{cases}$$

where [x] = greatest integer less than or equal to x.

Sol.

13. $f(x) = x.\left(\dfrac{e^{[x]+|x|} - 2}{[x]+|x|}\right)$, $x \neq 0$ & $f(0) = -1$ where [x] denotes greatest integer less than or equal to x. Test the differentiability of f(x) at x = 0.

Sol.

14. Discuss the continuity & the derivability in [0, 2]

of $f(x) = \begin{cases} |2x-3|[x] & \text{for } x \geq 1 \\ \sin\dfrac{\pi x}{2} & \text{for } x < 1 \end{cases}$

(where [$*$] denotes greatest integer function)

Sol.

15. Let f(x) = [3 + 4 sin x]. If sum of all the values of 'x' in [π, 2π] where f(x) fails to be differentiable, is $\dfrac{k\pi}{2}$, then find the value of k.

(where [$*$] denotes the greatest integer function)

Sol.

16. The function $f(x) = \begin{cases} ax(x-1)+b & \text{when } x < 1 \\ x-1 & \text{when } 1 \leq x \leq 3 \\ px^2+qx+2 & \text{when } x > 3 \end{cases}$.

Find the values of the constants a, b, p, q so that

(i) f(x) is continuous for all x

(ii) f '(1) does not exist

(iii) f '(x) is continuous at x = 3

Sol.

17. Examine the function, $f(x) = x.\dfrac{a^{1/x}-a^{-1/x}}{a^{1/x}+a^{-1/x}}$, $x \neq 0$

(a > 0) and f(0) = 0 for continuity and existence of the derivative at the origin.

Sol.

18. Discuss the continuity on $0 \leq x \leq 1$ & differentiability at x = 0 for the function.

$f(x) = x.\sin\dfrac{1}{x}\ \sin\dfrac{1}{x.\sin\dfrac{1}{x}}$ where $x \neq 0$, $x \neq 1/r\pi$ &

$f(0) = f(1/r\pi) = 0$, r = 1, 2, 3, ...

Sol.

19. $f(x) = \begin{cases} 1-x, & (0 \leq x \leq 1) \\ x+2, & (1 < x < 2) \\ 4-x, & (2 \leq x \leq 4) \end{cases}$ Discuss the continuity & differentiability of $y = f[f(x)]$ for $0 \leq x \leq 4$.

Sol.

20. Let f be a function that is differentiable every where and that has the following properties
(i) $f(x+h) = f(x) \cdot f(h)$ **(ii)** $f(x) > 0$ for all real x.
(iii) $f'(0) = -1$
Use the definition of derivative to find $f'(x)$ in terms of $f(x)$.
Sol.

21. Let f(x) be a function defined on (–a, a) with a > 0. Assume that f(x) is continuous at x = 0 and $\lim_{x \to 0} \frac{f(x) - f(kx)}{x} = \alpha$, where $k \in (0, 1)$ then compute $f'(0^+)$ and $f'(0^-)$, and comment upon the differentiability of f at x = 0.
Sol.

22. Consider the function, $f(x) = \begin{cases} x^2 \left|\cos\frac{\pi}{2x}\right| & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

(a) Show that $f'(0)$ exists and find its value
Sol.

(b) Show that $f'(1/3)$ does not exist.
Sol.

(c) For what values of x, $f'(x)$ fails to exist.
Sol.

23. Let f(x) be a real valued function not identically zero satisfies the equation, $f(x + y^n) = f(x) + (f(y))^n$ for all real x & y and $f'(0) \geq 0$ where n (> 1) is an odd natural number. Find f(10).
Sol.

24. A derivable function $f : R^+ \to R$ satisfies the condition $f(x) - f(y) \geq \ln(x/y) + x - y$ for every x, $y \in R^+$. If g denotes the derivative of f then compute the value of the sum $\sum_{n=1}^{100} g\left(\frac{1}{n}\right)$.
Sol.

EXERCISE – V JEE PROBLEMS

1. The function $f(x) = [x]^2 - [x^2]$ (where $[y]$ is the greatest integer less than or equal to y), is discontinuous at **[JEE 99,2]**
(A) all integers (B) all integers except 0 & 1
(C) all integers except 0 (D) all integers except 1
Sol.

2. Determine the constants a, b & c for which the function $f(x) = \begin{cases} (1+ax)^{1/x} & \text{for } x<0 \\ b & \text{for } x=0 \\ \dfrac{(x+c)^{1/3}-1}{(x+1)^{1/2}-1} & \text{for } x>0 \end{cases}$ is continuous at x = 0. **[REE 99,6]**
Sol.

3. Discuss the continuity of the function
$f(x) = \begin{cases} \dfrac{e^{1/(x-1)}-2}{e^{1/(x-1)}+2}, & x \neq 1 \\ 1, & x=1 \end{cases}$ at $x = 1$. **[REE 2001 (Mains), 3]**
Sol.

4. Let $f : R \to R$ be any function. Define $g : R \to R$ by $g(x) = | f(x) |$ for all x. Then g is **[JEE 2000(Scr.), 1]**
(A) onto if f is onto (B) one one if f is one one
(C) continuous if f is continuous
(D) differentiable if f is differentiable.
Sol.

5. Discuss the continuity and differentiability of the function, $f(x) = \begin{cases} \dfrac{x}{1+|x|}, & |x| \geq 1 \\ \dfrac{x}{1-|x|}, & |x| < 1 \end{cases}$. **[REE 2000, 3]**
Sol.

6. (a) Let $f : R \to R$ be a function defined by, $f(x) = \max [x, x^3]$. The set of all points where f(x) is NOT differentiable is **[JEE 2001 (Scr.)]**
(A) {–1, 1} (B) {–1, 0} (C) {0, 1} (D) {–1, 0, 1}
Sol.

(b) The left hand derivative of, $f(x) = [x] \sin (\pi x)$ at x = k, k an integer is
(where [$*$] denotes the greatest function)
(A) $(-1)^k(k - 1)\pi$ (B) $(-1)^{k-1}(k - 1)\pi$
(C) $(-1)^k k \pi$ (D) $(-1)^{k-1} k \pi$

Sol.

(c) Which of the following functions is differentiable at x = 0 ?

(A) cos (| x |) + | x | (B) cos (| x |) – | x |
(C) sin (| x |) + | x | (D) sin (| x |) – | x |

Sol.

7. Let $\alpha \in R$. Prove that a function $f : R \to R$ is differentiable at α if and only if there is a function $g : R \to R$ which is continuous at α and satisfies $f(x) - f(\alpha) = g(x)(x - \alpha)$ for all $x \in R$.
[JEE 2001 (mains), 5]

Sol.

8. The domain of the derivative of the function

$$f(x) = \begin{cases} \tan^{-1} x & \text{if } |x| \le 1 \\ \frac{1}{2}(|x|-1) & \text{if } |x| > 1 \end{cases}$$ is **[JEE 2002 (Scr.), 3]**

(A) R – {0} (B) R – {1}
(C) R – {–1} (D) R – {–1, 1}

Sol.

9. Let $f : R \to R$ be such that $f(1) = 3$ and $f'(1) = 6$.

The $\lim\limits_{x \to 0} \left(\frac{f(1+x)}{f(1)}\right)^{1/x}$ equals **[JEE 2002 (Scr.), 3]**

(A) 1 (B) $e^{1/2}$ (C) e^2 (D) e^3

Sol.

10. $f(x) = \begin{cases} x+a & \text{if } x<0 \\ |x-1| & \text{if } x \ge 0 \end{cases}$ and $g(x) = \begin{cases} x+1 & \text{if } x<0 \\ (x-1)^2+b & \text{if } x \ge 0 \end{cases}$

Where a and b are non negative real numbers. Determine the composite function gof. If (gof) (x) is continuous for all real x, determine the values of a and b. Further, for these values of a and b, is gof differentiable at x = 0 ? Justify your answer. **[JEE 2002, 5]**

Sol.

11. If a function $f : [-2a, 2a] \to R$ is an odd function such that $f(x) = f(2a - x)$ for $x \in [a, 2a]$ and the left hand derivative at x = a is 0 then find the left hand derivative at x = – a. **[JEE 2003 (Mains), 2]**

Sol.

12. (a) The function given by $y = ||x| - 1|$ is differentiable for all real numbers except the points
[JEE 2005 (Scr.), 3]

(A) {0, 1, –1} (B) ±1 (C) 1 (D) –1

Sol.

(b) If $|f(x_1)-f(x_2)| \le (x_1-x_2)^2$, for all $x_1, x_2 \in R$. Find the equation of tangent to the curve $y = f(x)$ at the point (1,2). **[JEE 2005 (Mains), 2]**

Sol.

13. If $f(x) = \min. (1, x^2, x^3)$, then **[JEE 2006, 5]**

(A) $f(x)$ is continuous $\forall x \in R$

(B) $f'(x) > 0, \forall x > 1$

(C) $f(x)$ is not differentiable but continuous $\forall x \in R$

(D) $f(x)$ is not differentiable for two values of x

Sol.

14. Let $g(x) = \dfrac{(x-1)^n}{\ln \cos^m(x-1)}$; $0 < x < 2$, m and n are integers $m \ne 0$, $n > 0$ and let p be the left hand derivative of $|x - 1|$ at $x = 1$. If $\lim\limits_{x \to 1^+} g(x) = p$, then **[JEE 2008, 3]**

(A) n = 1, m = 1 (B) n = 1, m = –1

(C) n = 2, m = 2 (D) n > 2, m = n

Sol.

15. Let $f : R \to R$ be a function such that $f(x + y) = f(x) + f(y), \forall x, y \in R$.

If $f(x)$ is differentiable at $x = 0$, then **[JEE 2011, 4]**

(A) $f(x)$ is differentiable only in a finite interval containing zero

(B) $f(x)$ is continuous $\forall x \in R$

(C) $f'(x)$ is constant $\forall x \in R$

(D) $f(x)$ is differentiable except at finitely many points

Sol.

16. Let $f : (0, 1) \to R$ be defined by $f(x) = \dfrac{b-x}{1-bx}$ where b is a constant such that $0 < b < 1$. Then

(A) f is not invertible on (0, 1) **[JEE 2011, 4]**

(B) $f \ne f^{-1}$ on (0, 1) and $f'(b) = \dfrac{1}{f'(0)}$

(C) $f = f^{-1}$ on (0, 1) and $f'(b) = \dfrac{1}{f'(0)}$

(D) f^{-1} is differentiable on (0, 1)

Sol.

17. If $f(x) = \begin{cases} -x-\dfrac{\pi}{2}, & x \le -\dfrac{\pi}{2} \\ -\cos x, & -\dfrac{\pi}{2} < x \le 0 \\ x-1, & 0 < x \le 1 \\ \ln x, & x > 1 \end{cases}$ then

(A) $f(x)$ is continuous at $x = -\dfrac{\pi}{2}$ **[JEE 2011, 4]**

(B) $f(x)$ is not differentiable at $x = 0$

(C) $f(x)$ is differentiable at $x = 1$

(D) $f(x)$ is differentiable at $x = -3/2$

Sol.

Answer Ex–I — SINGLE CORRECT (OBJECTIVE QUESTIONS)

1. A	**2.** B	**3.** D	**4.** C	**5.** C	**6.** B	**7.** B	**8.** B
9. B	**10.** B	**11.** B	**12.** D	**13.** B	**14.** B	**15.** D	**16.** D
17. C	**18.** B	**19.** D	**20.** C	**21.** D	**22.** B	**23.** B	**24.** C
25. D	**26.** C	**27.** C	**28.** C	**29.** C	**30.** D	**31.** C	**32.** D
33. C	**34.** D	**35.** D	**36.** B	**37.** B	**38.** C	**39.** C	**40.** B
41. D	**42.** D	**43.** D	**44.** B	**45.** C	**46.** D	**47.** A	**48.** B
49. C	**50.** B	**51.** B	**52.** C	**53.** A	**54.** B	**55.** A	**56.** D
57. A	**58.** D	**59.** C	**60.** B	**61.** A	**62.** D	**63.** D	**64.** D

Answer Ex–II — MULTIPLE CORRECT (OBJECTIVE QUESTIONS)

1. ABC	**2.** BCD	**3.** ABC	**4.** ABD	**5.** ABC	**6.** AC	**7.** ABC
8. AC	**9.** ABD	**10.** BD	**11.** ABD	**12.** ABD	**13.** AC	**14.** BC
15. CD	**16.** AC	**17.** AB	**18.** BD	**19.** BD	**20.** ABC	**21.** BD
22. ABD	**23.** ABD	**24.** ABD				

Answer Ex–III — SUBJECTIVE QUESTIONS

1. –1 **2.** a = 0, b = 1 **3.** $f(0^+) = -2$; $f(0^-) = 2$ hence f(0) not possible to define

4. **(a)** –2, 2, 3 **(b)** K = 5 **(c)** even **5.** $y_n(x)$ is continuous at x = 0 for all n and y(x) is dicontinuous at x = 0

6. f is cont. in $-1 \le x \le 1$ **7.** P not possible. **8.** **(a)** $4 - 3\sqrt{2} + a$, **(b)** a = 3

9. g(x) = 2 + x for $0 \le x \le 1$, 2 – x for $1 < x \le 2$, 4 – x for $2 < x \le 3$, g is discontinuous at x = 1 & x = 2

10. A = 1 ; f(2) = 1/2 **11.** a = 0 ; b = –1 **12.** a = 1/2, b = 4 **13.** a = – 3/2, b ≠ 0, c = 1/2

14. A = –4, B = 5, f(0) = 1 **15.** $f(0^+) = \frac{\pi}{2}$; $f(0^-) = \frac{\pi}{4\sqrt{2}} \Rightarrow$ f is discont. at x = 0 ;

$g(0^+) = g(0^-) = g(0) = \pi/2 \Rightarrow$ g is cont. at x = 0

16. the function f is continuous everywhere in [0, 2] except for x = 0, $\frac{1}{2}$, 1 & 2

17. discontinuous at all integral values in [–2, 2]

18. locus (a, b) → x, y is y = x – 3 excluding the points where y = 3 intersects it.

19. c = 1, a, b ∈ R **20.** 5 **21.** —

24. $k = 0$; $g(x) = \begin{cases} \ell n(\tan x) & \text{if } 0 < x < \frac{\pi}{4} \\ 0 & \text{if } \frac{\pi}{4} \le x < \frac{\pi}{2} \end{cases}$. Hence g(x) is continuous everywhere.

25. $g(x) = 4(x + 1)$ and limit $= -\frac{39}{4}$

27. $a = \frac{1}{\sqrt{2}}$, $g(0) = \frac{(\ell n 2)^2}{8}$

Answer Ex–IV | ADVANCED SUBJECTIVE QUESTIONS

1. f(x) is conti. but not derivable at $x = 0$

2. conti. $\forall\ x \in R$, not diff. at $x = 0, 1$ & 2

3. $0 < n \le 1$

4. conti. but not diff. at $x = 0$; diff. & conti. at $x = \pi/2$

5. conti. but not diff. at $x = 0$

7. f is cont. but not diff. at $x = 0$

8. $f'(1^+) = 3$, $f'(1^-) = -1$

9. $a = 1/2$, $b = 3/2$

10. not derivable at $x = 0$ & $x = 1$

11. f is cont. & derivable at $x = -1$ but f is neither cont. nor derivable at $x = 1$

12. discontinuous & not derivable at $x = 1$, continuous but not derivable at $x = 2$

13. not derivable at $x = 0$

14. f is conti. at $x = 1, 3/2$ & disconti. at $x = 2$, f is not diff. at $x = 1, 3/2, 2$

15. 24

16. $a \ne 1$, $b = 0$, $p = \frac{1}{3}$ and $q = -1$

17. If $a \in (0, 1)$ $f'(0^+) = -1$; $f'(0^-) = 1 \Rightarrow$ continuous but not derivable

If $a = 1$; $f(x) = 0$ which is constant $\Rightarrow$ continuous but not derivable

If $a > 1$ $f'(0^-) = -1$; $f'(0^+) = 1 \Rightarrow$ continuous but not derivable

18. conti. in $0 \le x \le 1$ & not diff. at $x = 0$

19. f is conti. but not diff. at $x = 1$, disconti. at $x = 2$ & $x = 3$. cont. & diff. at all other points

20. $f'(x) = -f(x)$

21. $f'(0) = \frac{\alpha}{1-k}$

22. **(a)** $f'(0) = 0$, **(b)** $f'\left(\frac{1^-}{3}\right) = -\frac{\pi}{2}$ and $f'\left(\frac{1^+}{3}\right) = \frac{\pi}{2}$, **(c)** $x = \frac{1}{2n+1}$ $n \in I$

23. $f(x) = x \Rightarrow f(10) = 10$

24. 5150

Answer Ex–V | JEE PROBLEMS

1. D

2. $a = \ln\frac{2}{3}$; $b = \frac{2}{3}$; $c = 1$

3. Discontinuous at $x = 1$; $f(1^+) = 1$ and $f(1^-) = -1$

4. C

5. Discont. Hence hot deri. at $x = 1$ & -1. Cont. & deri. at $x = 0$

6. **(a)** D, **(b)** A, **(c)** D

8. D

9. C

10. $a = 1$; $b = 0$ $(gof)'(0) = 0$

11. $f'(a^-) = 0$

12. **(a)** A, **(b)** $y - 2 = 0$

13. A, C

14. C

15. B,C

16. C,D

17. A,B,C,D

Solution Book

Continuity & Diffrentiability

JEE Main | CBSE

CONTINUITY & DIFFERENTIABILITY

EXERCISE – I HINTS & SOLUTIONS

Sol.1 **A**

$$f(x) = \begin{cases} \dfrac{\cos(\sin x) - \cos x}{x^2}, & x \neq 0 \\ a & , x = 0 \end{cases}$$

for continuity, $\lim\limits_{x \to 0} f(x) = f(0)$

$$\lim_{x \to 0} \frac{\cos(\sin x) - \cos x}{x^2} = a$$

$$\lim_{x \to 0} \frac{2\sin\left(\dfrac{x - \sin x}{2}\right)\sin\left(\dfrac{\sin x + x}{2}\right)}{x^2} = a$$

$$\lim_{x \to 0} \frac{2\sin\left(\frac{x - \sin x}{2}\right)}{\left(\frac{x - \sin x}{2}\right)} \times \frac{\sin\left(\frac{x + \sin x}{2}\right)}{\left(\frac{x + \sin x}{2}\right)} \times \left(\frac{x^2 - \sin^2 x}{4}\right) = a$$

$$2 \times \frac{0}{4} = a \Rightarrow a = 0$$

Sol.2 **B**

$$f(x) = \begin{cases} \dfrac{\sqrt{1+px} - \sqrt{1-px}}{x} & , -1 \le x \le 0 \\ \dfrac{2x+1}{x+2} & , 0 \le x \le 1 \end{cases}$$

since it is cont, so,

$$\lim_{x \to 0^-} f(x) = f(0)$$

$$\lim_{h \to 0} \frac{\sqrt{1+p(-h)} - \sqrt{1-p(-h)}}{-h} = -\frac{1}{2}$$

$$\lim_{h \to 0} \frac{(1-ph) - (1+ph)}{-h\left\{\sqrt{1-ph} + \sqrt{1+ph}\right\}} = -\frac{1}{2}$$

$$\frac{+2p}{2} = -\frac{1}{2}$$

$$p = -1/2$$

Sol.3 **D**

$$f(x) = \left|\left(x + \frac{1}{x}\right)[x]\right|, \quad x \in [-2, 2]$$

$$\lim_{x \to 2^-} f(x) = \lim_{h \to 0} \left|\left(\frac{5}{2} - h\right)1\right| = \frac{5}{2}$$

$$f(2) = \left|\frac{5}{2} \times 2\right| = 5$$

so, disc at x = 2

now defining function

$$f(x) = \left|\left(x + \frac{1}{2}\right)[x]\right| = \begin{cases} 3 & ; \ -2 \le x < -1 \\ \frac{1}{2} & ; \ -1 \le x < 0 \\ 0 & ; \ 0 \le x < 1 \\ \frac{3}{2} & ; \ 1 \le x \le 2 \\ 2 & ; \ 2 \le x < 3 \end{cases}$$

by defining the function we can say that this it is disc at x = 0

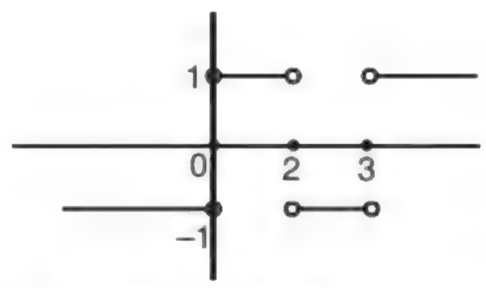
Sol.4 **C**

$f(x) = \text{sgn}\ (x),\ g(x) = x(x^2 - 5x + 6)$

$f(g(x)) = \text{sgn}\ (x(x^2 - 5x + 6))$

$\quad\quad = \text{sgn}\ (x(x - 2)\ (x - 3))$

$$g(g(x)) = \begin{cases} 1 & ; \ x\,(x-2)\,(x-3) > 0 \\ & \quad x \in (0, 2) \cup (3, \infty) \\ 0 & ; \ x(x-2)(x-3) = 0 \\ & \quad x = 0, 2, 3 \\ -1 & ; \ x(x-2)(x-3) < 0 \\ & \quad x \in (-\infty, 0) \cup (2, 3) \end{cases}$$



so, f(g(x)) is disc. at exactly points 0, 2 & 3

Sol.5 **C**

$$y = \frac{1}{t^2 + t - 2},\ t = \frac{1}{x - 1}$$

$$y = \frac{1}{\dfrac{1}{(x-1)^2} + \dfrac{1}{x-1} - 2}$$

$$y = \frac{(x-1)^2}{1+(x-1)-2(x-1)^2}$$

$$y = \frac{x^2-2x+1}{x-2x^2-2+4x} = \frac{x^2-2x+1}{-2x^2+5x-2}$$

$$y = \frac{(x-1)^2}{-2x^2+4x+x-2} = \frac{(x-1)^2}{-2x(x-2)+1(x-2)}$$

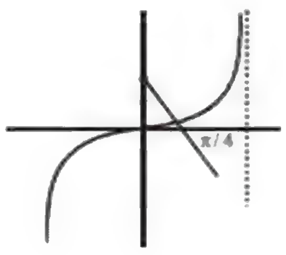
$$y = \frac{(x-1)^2}{(x-2)(-2x+1)}$$

by $\Rightarrow x \in R - \left\{2, +\frac{1}{2}\right\}$ so disc. at 1/2 & 2 let we also include x = 1 because at x = 1 't' is not defined.

Sol.6 **B**

$2 \tan x + 5x - 2 = 0$

$\tan x = -\dfrac{5x}{2} + 1$



Sol.7 **B**

$f(x) = x\left(\sqrt{x} - \sqrt{x+1}\right)$

$$f(0^+) = \lim_{h\to 0} \frac{h\left(\sqrt{h}-\sqrt{h+1}\right)}{h}$$

$$= \lim_{h\to 0} \frac{h-h-1}{\sqrt{h}+\sqrt{h+1}} = -1$$

Sol.8 **B**

$$f(x) = \begin{cases} x\dfrac{(3e^{1/x}+4)}{2-e^{1/x}} & , \; x \neq 0 \\ 0 & , \; x = 0 \end{cases}$$

$$f'(0^+) = \lim_{h\to 0} \frac{h\dfrac{(3e^{1/h}+4)}{2-e^{1/h}} - 0}{h}$$

$$= \lim_{h\to 0} \frac{3\left(1+\dfrac{1}{h}\right)+4}{2-1-\dfrac{1}{h}}$$

$$= \lim_{h\to 0} \frac{7h+3}{h-1} = 3$$

$$f'(0^-) = \lim_{h\to 0} \frac{-h\dfrac{(3e^{-1/h}+4)}{2-e^{-1/h}} - 0}{-h}$$

$$= \lim_{h\to 0} \frac{3\left(1-\dfrac{1}{h}\right)+4}{2-\left(1-\dfrac{1}{h}\right)}$$

$$= \frac{7h-3}{h+1} = -3$$

so, not diff. at x = 0

Sol.9 **B**

$$f(x) = \frac{x}{\sqrt{x+1}-\sqrt{x}} = \frac{x(\sqrt{x+1}+\sqrt{x})}{x+1-x}$$

$f(x) = x\left(\sqrt{x+1}+\sqrt{x}\right)$

Now, RHD

$$f(0^+) = \lim_{h\to 0} = \frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0} \frac{h(\sqrt{h+1}-\sqrt{h})-0}{h}$$

$= 1$

since ve⁻ values are not in domain of f(x) hance diff will lie diudid by RHD. Since RHD is finit hence f(x) is diff.

Sol.10 **B**

$f(x) = \sin^{-1}(\cos x)$

$$f(0^+) = \lim_{h\to 0} \frac{\sin^{-1}(\cosh) - \dfrac{\pi}{2}}{h}$$

$$= \lim_{h\to 0} \frac{\sin^{-1}\sin\left(\dfrac{\pi}{2}-h\right) - \dfrac{\pi}{2}}{h}$$

$$f(0^+) = \lim_{h\to 0} \frac{\dfrac{\pi}{2}-h-\dfrac{\pi}{2}}{h} = -1$$

$$f'(0^-) = \lim_{h\to 0} \frac{\sin^{-1}(\cos(-h)) - \dfrac{\pi}{2}}{-h}$$

$$= \lim_{h\to 0} \frac{\sin^{-1}(\cosh) - \dfrac{\pi}{2}}{-h}$$

$$= \lim_{h\to 0} \frac{\frac{\pi}{2} - h - \frac{\pi}{2}}{-h} = 1$$

Sol.11 B

$$f(x) = \begin{cases} \dfrac{x^2-1}{x^2+1} & , \ 1 < x \le 2 \\ \dfrac{x^3 - x^2}{4} & , \ 2 < x \le 3 \\ \dfrac{9}{4}(|x-4| + |2-x| & , \ 3 < x < 4 \end{cases}$$

$$f'(2^+) = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{(2+h)^2\{2+h-1\}}{4} - \frac{3}{5}}{h}$$

$$= \lim_{h\to 0} \frac{5(h^3 + 4h^2 + 4h + h^2 + 4h + 4) - 12}{20h}$$

$$= \lim_{h\to 0} \frac{5(h^3 + 5h^2 + 8h).+8}{20h} = \text{Not exists}$$

Hence f(x) is not diff at x = 2

$$f'(3^+) = \lim_{h\to 0} \frac{f(3+h) - f(3)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{9}{4}(|h-1| + |2-3-h|) - \frac{18}{4}}{h}$$

$$= \frac{9}{4} \lim_{h\to 0} \frac{-h+1+1+h-2}{h} = 0$$

$$f'(3^-) = \lim_{h\to 0} \frac{f(3-h) - f(3)}{-h}$$

$$= \lim_{h\to 0} \frac{\frac{(3-h)^3 - (3-h)^2}{4} - \frac{18}{4}}{h}$$

$$= \frac{1}{4} \lim_{h\to 0} \frac{(3-h)^2(3-h-1) - 18}{h}$$

$$= \frac{1}{4} \lim_{h\to 0} \frac{18 - 9h + 2h^2 - h^3 - 12h + 6h^2 - 18}{h}$$

$$= \frac{1}{4} \lim_{h\to 0} -9 - 12 = -\frac{21}{4}$$

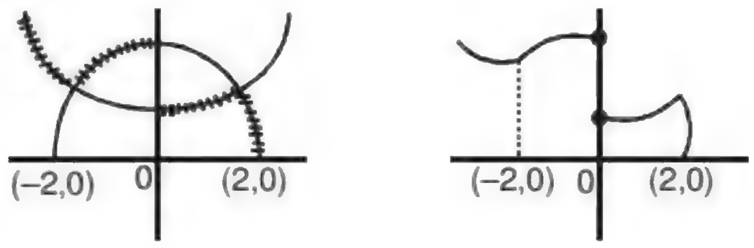
since $f'(3^+) \ne f'(3^-)$

Hence f(x) is now diff at x = 3.

Sol.12 D

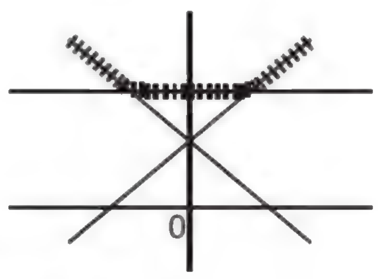
$$f(x) = \begin{cases} \max(\sqrt{4-x^2}, \sqrt{1+x^2}) & -2 \le x \le 0 \\ \min(\sqrt{4-x^2}\ \sqrt{1+x^2}) & 0 < x \le 2 \end{cases}$$

$y = 4 - x^2$, $y = 1 + x^2$



Sol.13 B

f(x) = max {a – x, a + x, b}



so not diff. at two points

Sol.14 B

If f is differentiable everywhere.
then |f| will also be diff. everywhere.
and if two fns. are diff. then sum of then will also be diff. everywhere

Sol.15 D

$f(x + y) = f(x) . f(y), f(3) = 3$

$f'(0) = 11, f(3) = ?$

$$f(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x)\cdot f(h) - f(x)}{h}$$

$$= f(x) \cdot \lim_{h\to 0} \frac{f(h) - 1}{(h)}$$

$$f'(3) = f(3) \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$$

$f'(3) = f(3) . f'(0)$

$f'(3) = 3 \times 11 = 33$

$[\because\ f(0) = f(0) . f(0) \Rightarrow f(0) = 1]$

Sol.16 D

$f(x + 2y) = f(x) + f(2y) + 2xy$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{f(x)+f(h)-f(x)+2xy}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{f(0+h)-(0)}{h} + 2x$$

$f'(x) = f'(0) + 2$

Sol.17 C

$f(x) = x - x^2$

$$g(x) = \begin{cases} \max f(t), & 0 \le t \le x,\ 0 \le x \le 1 \\ \sin \pi x, & x > 1 \end{cases}$$

max f(t) will be obtained when t = x. so
$\max (f(t)) = x - x^2$

$$\text{so} \quad f'(1^+) = \lim_{h\to 0} \frac{\sin \pi(1+h) - 0}{h}$$

$$= \lim_{h\to 0} -\frac{\sin \pi h}{\pi h}\pi = -\pi$$

$$f'(1^-) = \lim_{h\to 0} \frac{(1-h)^2-(1-h)^2-0}{-h}$$

Not diff. at x = 1 but cont.

Sol.18 B

$g(x) = x - [x]\ f(0) = f(1)$
$h(x) = f(g(x))$
Let $x = a \in I$

$$h(a^+) = \lim_{x\to a^+} f(\{x\}) = f(0)$$

$$h(a^-) = \lim_{x\to a^-} f(g(x)) = f(1)$$

$h(a^+) = h(a^-)$ hence then is cont.

Sol.19 D

$$f(x) = \begin{cases} \log_{(4x-3)}(x^2-2x+5) & ; \ \frac{3}{4} < x < 1 \ \&\ x > 1 \\ 4 & x = 1 \end{cases}$$

$$\text{LHS } f(1^-) = \lim_{h\to 0} \log_{1-3h} \{(1-h)^2 - 2(1-h) + 5\}$$

$$= \lim_{h\to 0} \log_{(1-3h)} \{h^2 + 1 - 2h - 2 + 2h + 5\}$$

$$= \lim_{h\to 0} \log_{(1-3h)} (h^2 + 4)$$

$$= \lim_{h\to 0} \log (h^2+4) \times \frac{-3h}{\log(1-3h)} \times \frac{1}{-3h} = \infty$$

similarly $f(1^+)$ will be ∞.

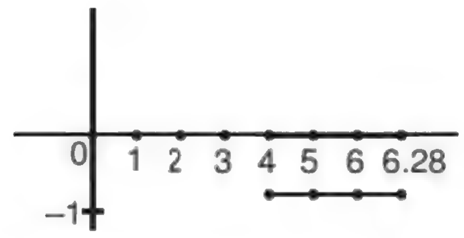
Sol.20 C

$f(x) = x^2, \quad x \in Q^c$
$\quad = 1, \quad x \in Q$
By short trick
$x^2 = 1 \Rightarrow x = \pm 1$
Hence f(x) will be const. at $x = \pm 1$

Sol.21 C

$f(x) = [\sin [x]]$
we'll define the given function as follows :-

$$[\sin [x]] = \begin{cases} 0 & ; \ 0 \le x < 1 \\ 0 & ; \ 1 \le x < 2 \\ 0 & ; \ 2 \le x < 3 \\ 0 & ; \ 3 \le x < 4 \\ -1 & ; \ 4 \le x < 5 \\ -1 & ; \ 5 \le x < 6 \end{cases}$$



so point where function is not cont. is (4, –1)

Sol.22 B

$$f(x) = \lim_{t\to\infty} \left\{ \frac{(1+\sin \pi x)^t - 1}{(1+\sin \pi x)^t + 1} \right\}$$

$$\lim_{t\to 0} \frac{(1+\sin\pi x)^t - 1}{(1+\sin \pi x)^t + 1} \begin{cases} 0 & \begin{array}{c} 1+\sin\pi x = 1 \\ \sin \pi x = 0 \\ \pi x = n\pi \end{array} \\ -1 & \begin{array}{c} x = n, n = 0,1,2,\ldots\ldots \\ 1+\sin \pi x > 1 \\ \sin \pi x > \sin 0 \\ x > n \end{array} \\ -1 & \begin{array}{c} 1 > 1+\sin\pi x > 0 \\ \pi x > -\frac{\pi}{2} \\ 0 > x > -\frac{n}{2} \end{array} \end{cases}$$

$$\text{Now } f(x) = \begin{cases} 0 & x = n, n = 1,2,3\ldots \\ -1 & x > n \\ -1 & -\frac{n}{2} < n < 0 \end{cases}$$

(i)	$f(0^+) = -1$	(ii)	$f(1^+) = -1$
	$f(0^+) = -1$		$f(1^-) = -1$
	$f(0) = 0$		$f(1) = 0$

Similarly for all integer the function will the disc.

Sol.23 B

$$f(x) = \begin{cases} \sqrt{x}\left(1+\sin\frac{1}{x}\right), & x>0 \\ -\sqrt{x}\left(1+\sin\frac{1}{x}\right), & x<0 \\ 0, & x=0 \end{cases}$$

$$f'(0^+) = \frac{f(0+h)-f(0)}{h} \Rightarrow \lim_{h\to0} d$$

$$= \lim_{h\to0} \frac{\left(1+\sin\frac{1}{h}\right)}{\sqrt{h}} = \text{m.d.} \Rightarrow \text{N. diff.}$$

$$f(0^+) = \lim_{h\to0} \sqrt{h}\left(1+\sin\frac{1}{h}\right)$$

$$= \lim_{h\to0} \frac{h}{\sqrt{h}}\left(1+\sin\frac{1}{h}\right)$$

$$= \lim_{h\to0} \left(\frac{1}{\sqrt{h}}\right)\frac{\left(1+\sin\frac{1}{h}\right)}{(1/h)}$$

$$= \lim_{h\to0} \frac{1}{\sqrt{h}}\left\{h + h\sin\frac{1}{h}\right\}$$

$= 0$

$$f(0^-) = \lim_{h\to0} -\sqrt{h}\left(1-\sin\frac{1}{h}\right)$$

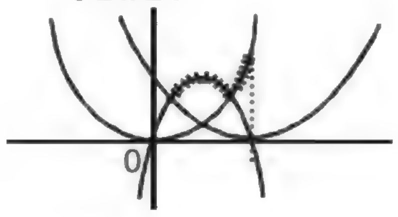
$= 0$

Hence (B)

Sol.24 C

$f(x) = \max\{x^2, (x-1)^2, 2x(1-x)\}$

$0 \le x \le 1$



so, (c)

Sol.25 D

$[n + p\sin x] = n[p\sin x]$

$\therefore$ $[p\sin x]$ is non. diff. where $p\sin x$ is as integer but P is prime and $0 < \sin x \le 1$ $[0 < x < \pi]$

$\therefore$ $p\sin x$ is an integer only when

$\sin x = \frac{r}{p}$; where $0 < r \le p$ and $r \in N$

For $r = p$; $\sin x = 1 \Rightarrow x = \frac{\pi}{2}$ in $(0, \pi)$

For $0 < r < p$; $\sin x = \frac{r}{p}$

$x = \sin^{-1}\left(\frac{r}{p}\right)$ or $\pi - \sin^{-1}\left(\frac{r}{p}\right)$

Number of such values of $x = P - 1 + P - 1 = 2P - 2$

Total No. of points $= 2P - 2 + 1 = 2P - 1$

Sol.26 C

$f(x) = x^3 - x^2 + x + 1$

$$g(x) = \begin{cases} \max(f(t))\,; 0 \le t \le x & \text{for } 0 \le x \le 1 \\ x^2 - x + 3\,; 1 < x \le 2 \end{cases}$$

max {f(t)} will be obtained when 't' would be max.

so, $t = x$.

so, $\max\{f(t)\} = x^3 - x^2 + x + 1$

$$f'(1^+) = \lim_{h\to0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to0}\frac{(1+h)^2-(1+h)+3-2}{h}$$

= not defined

so not derivable

Now check cont by,

$$f(1^+) = \lim_{h\to0} f(1+h)$$

$$= \lim_{h\to0} (1+h)^2 - (1+h) + 3$$

$= 3$

& $f(1) = 2$ so $f(x)$ is not cont.

Sol.27 C

By using L' Hospital rule

$$= \lim_{x\to0}\frac{2f'(x) - f'(2x) + 4f'(4x)}{2x}$$

Again

$$= \lim_{x\to0}\frac{2f''(x) - 12f''(2x) + 16f''(4x)}{2} = 12$$

Sol.28 C

Put $y = 0 \Rightarrow f\left(\frac{x}{3}\right) = \frac{f(x)}{3} \Rightarrow f(3x) = 3f(x)$

$$f'(x) = \lim_{h\to0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{f(3x)+f(3h)}{3} - f(x)}{h} = \lim_{h\to 0} \frac{\frac{8f(x)+3f(x)}{3} - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(h)-f(0)}{h} = f'(0)$$

$f'(x) = 3 \Rightarrow f(x) = 3x + c \Rightarrow f(x) = 3x$

Sol.29 C

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x)+f(h)+xh-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(h)}{h} + x$$

$f'(x) = x + 3$

$f(x) = \frac{x^2}{2} + 3x + c \qquad f(0) = 0$

$f(x) = \frac{x^2}{2} + 3x \qquad \Rightarrow c = 0$

Sol.30 D

Put $x = 0, y = 0 \Rightarrow f(0) = \frac{4}{7}$

Now put $y = 0$

$$f\left(\frac{x}{3}\right) = \frac{4-2[f(x)+f(0)]}{3}$$

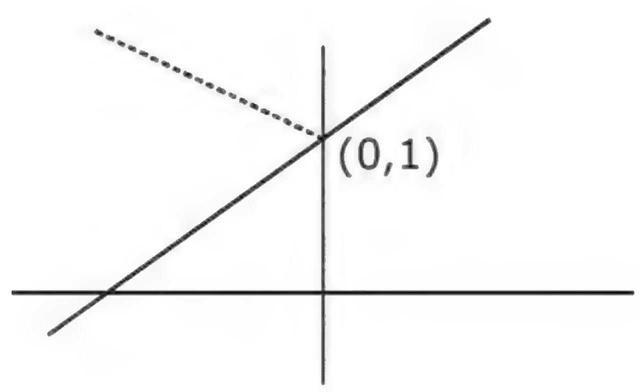
$\Rightarrow 3f(x) = 4 - 2\,[f(3x) + 6(0)]$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

Now proceed as in question (28)

$f(x) = \frac{4}{7}$

Sol.31 C



Sol.32 D

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\left(\frac{1}{x} - \frac{2}{e^{2x}-1}\right)$$

$$= \lim_{x\to 0}\left(\frac{e^{2x}-1-2x}{x(3^{2x}-1)}\right)$$

$$= \lim_{x\to 0} \frac{e^{2x}-1-2x}{2x^2} = 1$$

Sol.33 C

$$\frac{|f(x)-f(y)|}{x-y} \le (x-y)$$

$$\lim_{x\to y} \frac{|f(x)-f(y)|}{x-y} \le \lim_{x\to y} (x-y)$$

$f'(x) = 0 \Rightarrow f(x) = C \Rightarrow f(0) = 0 \Rightarrow C = 0$

$f(1) = 0$

Sol.34 D

$f(x) = |x-1| + |x-2| + \cos x$

All three fns are cont. is [0, 4]

so sum of all these functions is also a cont. funs.

Sol.35 D

$g\left(\frac{1}{2}\right) = f(1) = 0$

$f\left(\frac{1^+}{2}\right) = f[1^+] = f(1) = 0$

$g\left(\frac{1^-}{2}\right) = f[0] = f(0) = 1$

Discont. at $x = \frac{1}{2}$

Sol.36 B

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x).f(h)-f(x)}{h} = f(x)\lim_{h\to 0} \frac{f(h)-f(0)}{h}$$

$f'(x) = f(x)\,.\,f'(0)$

$f'(x) = 2f(x)$

$\ln f(x) = 2x + C \quad : C = 0$

$f(x) = p^{2x}$

Sol.37 B

$$f'(x) = \lim_{h\to 0} \frac{f(x+n)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x).f(h)-f(x)}{h} = f(x) \lim_{h\to 0} \frac{f(h)-f(0)}{h}$$

$f'(x) = f(x) \,.\, f'(0)$

$f'(x) = 2f(x)$

$\ln f(x) = 2x + C \quad : C = 0$

$f(x) = e^{2x}$

Sol.38 C

By doing rationlize

$$f(0) = \lim_{x\to 0} \frac{(a^2-ax+x^2)-(a^2+ax+x^2)}{(x+x)-(a-x)}$$

$$\times \frac{(\sqrt{a+x}+\sqrt{a-x})}{(\sqrt{a^2-ax+x^2}+\sqrt{a^2+ax+x^2})}$$

$$= -\frac{-2ax}{2x}\left(\frac{2\sqrt{a}}{2a}\right)$$

$f(0) = -\sqrt{a}$

Sol.39 C

$$\text{RHL} = \lim_{x\to \frac{\pi^+}{2}} f(x) = \lim_{h\to 0} \frac{\sin\left\{\cos\left(\frac{\pi}{2}+h\right)\right\}}{h}$$

$$= \lim_{h\to 0} \frac{\sin\{-\sin b\}}{h} = \lim_{h\to 0} \frac{\sin(1-\sin h)}{h} \to \infty$$

DNE

Sol.40 B

$$f(x) = \left[\sqrt{2}\sin\left(\frac{\pi}{4}+h\right)\right]$$

$$-\frac{1}{\sqrt{2}} \qquad 0 \qquad \frac{1}{\sqrt{2}}$$

$$\Downarrow \qquad \Downarrow \qquad \Downarrow$$

$$(2) \qquad (1) \qquad (2)$$

Total solutions = 5

Sol.41 D

$$f(1^+) = \lim_{x\to 1^+} x^2\left[\frac{1}{x^2}\right] = 0$$

$$f(1^-) = \lim_{x\to 1^-} x^2\left[\frac{1}{x^2}\right] = 1$$

Discont. at $x = 1$

similarly for $x = -1$

$$f(x) = x^2\left(\frac{1}{x^2}-\left\{\frac{1}{x^2}\right\}\right) = 1 - x^2\left\{\frac{1}{x^2}\right\}$$

$$f(0^+) = \lim_{x\to 0^+} 1 - x^2\left\{\frac{1}{x^2}\right\} = 1$$

$f(0^{-1}) = 1$ But $f(0) = 0$

So dicont. at $x = 0$

at $x = 2$, RHL = LHL = $f(2) = 0$

coint. at $x = 2$

Sol.42 D

$$\text{RHL} = \lim_{h\to 0} \sin[\ell nh] = [-1, 1]$$

$$\text{LHL} = \lim_{h\to 0} \sin[\ell n\, h] = [-1, 1]$$

So DNE

Sol.43 D

$$f(x) = \frac{|x-3|}{|x-2|} + \frac{1}{1+[x]}$$

$x \neq 2$ $\qquad 1 + [x] = 0$

$[x] \neq -1,$ $\qquad x \in [1, 0)$

And $[x]$ will be disoint. at every integer

So $x \in R - \{(-1, 0) \cup n, n \in I\}$

Sol.44 B

f(x) should be a constant function.

Sol.45 C

$$\text{RHL} = \lim_{h\to 0} \frac{a^h-1}{h} = \ell na$$

$$\text{LHL} = \lim_{h\to 0} \frac{a^{-1-h}-1}{-1-h} = 1 - \frac{1}{a}$$

$f(x) = \ell na \quad \Rightarrow$ Discont. at $x = 0$

Sol.46 D

$f'(0^+) = p + q$ $\qquad$ ....(1)

$f'(0^-) = -p + q$ $\qquad$ ....(2)

$f'(0^+) = f'(0^-) \Rightarrow p + q = 0, r \in R$

Sol.47 A

$g(x) = [x] + 1$

$h(x) = g(\sin x) = [\sin x] + 1$

$[\sin x]$ is discont at $x = \frac{\pi}{2}$

$\Rightarrow$ $[\sin x] + 1$ is also a discont at $x = \frac{\pi}{2}$

Sol.48 B

$f(x) = [\tan^2 x]$

RHL = $\lim_{x\to 0^+} [\tan^2 x] = 0$

LHL = $\lim_{x\to 0^-} [\tan^2 x] = 0$: $f(x) = 0$

So cont. at $x = 0$

Sol.49 C

$f(x) = [x]^2 + \sqrt{(x-[x])^2}$

DIscont. at every integer because $[x]$ is discont. at every integer.

But $f(x)$ is cont. at $x = 1$

So option (C) is correct.

Sol.50 B

$f'(0^+) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \ell \neq 0$

$f'(0^-) = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h} = -\ell \neq 0$

Non. diff. But cont. at $x = 0$

Sol.51 B

$f(x) = \frac{3-2\sqrt{3}+2x-x^2}{x-\sqrt{3}}$

$= \frac{(\sqrt{3}-x)(\sqrt{3}+x)+2(x-\sqrt{3})}{x-\sqrt{3}}$

$f(x) = 2 - (\sqrt{3} + x)$

$f(\sqrt{3}) = 2 - 2\sqrt{3} = 2(1 - \sqrt{3})$

Sol.52 C

$f(x) = \text{Sgn}(4 - 2\sin^2 x - 2\sin x)$

$= \text{Sgn}[(\sin x + 2)(2 - 2\sin x)]$

$f(x) = 0$ when $x > \frac{\pi}{2}$

$= 1$ $x < \frac{\pi}{2}$

$= -1$ $\sin x > 1$ not possible

SO isolated pt. discont.

Sol.53 A

RHL $\Rightarrow x = 0+h$

$\lim_{h\to 0} |g(f(h))|$

as $h \to 0$ $f(h) \to$; $g(0) \to 0$

RHL = 0

$h(0) = 0$

So cont. at $x = 0$

Sol.54 B

$f(1) = 0$

$f(x) = \begin{cases} 1 & x > 1 \\ 0 & x < 1 \end{cases}$

Discont. at $x = 1$

Sol.55 A

RHL = $\lim_{x\to 0^+} \frac{[\{|x|\}]e^{x^2}\{[x+\{x\}]\}}{(e^{1/x^2}-1)\text{Sgn}(\sin x)}$

fraction part of greatest integer in always zero.

So RHL = LHL = 0

So coint. at $x = 0$

Sol.56 D

RHL = $\lim_{x\to 0^+} \frac{\ln(e^{x^2}+2\sqrt{x}+1-1)(e^{x^2}+2\sqrt{x}-1)}{(e^{x^2}+2\sqrt{x}-1)\sqrt{x}}$

$= 2$

LHL = $\lim_{x\to 0^-} \frac{x[x]^2 \ell n2}{\ell n(x+1)} = \ell n2$

Non-Removable disont at $x = 0$

Sol.57 A

RHL = $\lim_{x\to 0^+} \frac{\sqrt{1+x}-\sqrt{1-x}}{x} = 1$(Rationlize)

LHL = $\frac{1}{\sqrt{2}} f(g(x))$

$= \lim_{x\to 0^-} \frac{1}{\sqrt{2}} \frac{|\sqrt{2}\cos x| - |\sqrt{2}\sin x|}{\cos 2x}$

$$= \lim_{x \to 0^-} \frac{1}{\cos x - \sin x} = 1$$

cont. at x = 0

Sol.58 D

$$\lim_{x \to 0} \frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - ....\right) - x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + ...\right)}{x^2}$$

$= 0$

f(x) is cont at x = 0

Sol.59 C

$$\text{RHD} = \lim_{h \to 0} \frac{\frac{h}{\sinh} - 1}{h} = \lim_{h \to 0} \frac{h - \sinh}{h^2} = 0$$

$$\text{LHD} = \lim_{h \to 0} \frac{\frac{-h}{\sinh} - 1}{h} = \lim_{h \to 0} \frac{-h - \sinh}{h^2} \to \infty$$

Non. diff. at x = 0

RHL = 1 Discont.

LHL = –1

Sol.60 B

$$f(x) = \begin{cases} x.\dfrac{a^{-2|x|} - 5}{3 + a^{1/|x|}} & ; |x| \neq 0; a > 1 \\ 0 & ; x = 0 \end{cases}$$

$f'(0^+) = 0$; $f'(0^-) = 0$

diff. & cont. at x = 0

Sol.61 A

LHD (x = 1) = RHD (x = 1)

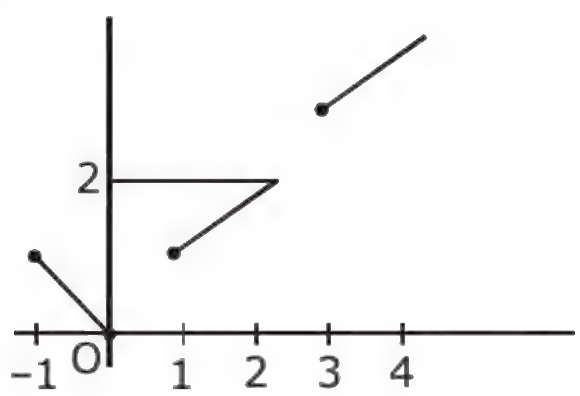
1 = 2a + b ....(1)

LHL (x = 1) RHL (x = 1)

1 = a + b + c ....(2)

b = 1-2a, c = a

Sol.62 D



Discont at x = 1, 2, 3

Non. diff. at x = 1, 2, 3

Sol.63 D

RHD (at x = 0) = 0; LHD = 1

RHD (at x = 1) = 2; LHD = 2

RHL (at x = 0) = 0 = LHL

RHL (at x = 1) = LHL (x = 1)

Diff. and cot. at x = 1

Non diff. x = 0 but cont. at x = 0

Sol.64 D

$$\text{RHD} = \lim_{h \to 0} \frac{h + h + h\sinh - 0}{h} = \lim_{h \to 0} 2 + \sin h = 2$$

$$\text{LHD} = \lim_{h \to 0} \frac{-h + (1 - h) - h\sin(1 - h)}{-h} \to \infty$$

Non. diff. at x = 0

$$\text{RHL} = \lim_{h \to 0} h + h + h \sin h = 0$$

$$\text{LHL} = \lim_{h \to 0} -h + 1 - h + h \sin(1 - h) = h$$

discont at n = 0

EXERCISE – II HINTS & SOLUTIONS

Sol.1 **A,B,C**

(A) $f(x) = \dfrac{1}{1+2^{1/x}}$ **(B)** Not defined at x = 0

RHL = 0 RHL $= -\dfrac{\pi}{2}$

LHL = 1 LHL $= \dfrac{\pi}{2}$

(C) $f(x) = \dfrac{e^{1/x}-1}{e^{1/x}+1}$ **(D)** $f(x) = \dfrac{1}{\ell n|x|}$

RHL = 1 ; LHL = –1

Sol.2 **B,C,D**

(B) $x = 1 - x \Rightarrow x = 1/2$

(C) $x = 0$

(D) $x = -x \Rightarrow x = 0$

Sol.3 **A,B,C**

$\lim\limits_{x\to1^+} |x-3| = 2$

$\lim\limits_{x\to1^-} \left(\dfrac{x^2}{4}\right) - \left(\dfrac{3x}{2}\right) + \dfrac{13}{4} = 2$

function is cont. at x = 1

function is also diff. at x = 1

and will be cont. at x = 3

Sol.4 **A,B,D**

tan x will be discontinuous at $x = \dfrac{\pi}{2}$

and $|x-0.5|$ and $|x-1|$ will be non-differentiable at x = 0.5 and x = 1 respectively.

so non diff. at $x = \dfrac{1}{2}, 1, \dfrac{\pi}{2}$

Sol.5 **A,B,C**

$f(x) = [x]$; $g(x) = \begin{cases} 0 & , x\in I \\ x^2 & , x\in R-I \end{cases}$

for g(x)

RHL (at x = 1) = 1 g(1) = 0

LHL (at x = 1) = 1

so discont. at x = 1 (A)

for f(x)

RHL (at x = 1) = 1

LHL (at x = 1) = 0 dis cont. at x = 1 (B)

For gof(x)

$gof(x) = [x]^2$, $\lim\limits_{x\to1} [1]^2 = 1$ cont. $\forall\ x \in R$

Sol.6 **B,D**

$$f(x) = \begin{cases} 3 - \left[\cot^{-1}\left(\dfrac{2x^3-3}{x^2}\right)\right] & x>0 \\ \{x^2\}\cos(e^{1/x}) \quad \text{for} & x<0 \end{cases}$$

RHL $\lim\limits_{x\to0^+} 3 - \left[\cot^{-1}\dfrac{2x^2-3}{x^2}\right] = 3-3 = 0$

$\cot^{-1}(-\infty) \to [\pi] = 3$

LHL $\lim\limits_{x\to0^-} \{x^2\}\cos e^{1/x}$

x = 0 – h

$$\lim_{h\to0} \underset{\underset{0}{\downarrow}}{\{h^2\}} \times \underbrace{\cos e^{-1/h}}_{\underset{1}{\downarrow}} = 0$$

Sol.7 **A,B,C**

$f(x) = [x] + \sqrt{x-[x]}$

$f(x) = [x] + \sqrt{\{x\}} \Rightarrow x - \{x\} + |\{x\}|$

$f(x) = x$

f(x) is cont. on R, R^+, R – I

Sol.8 **A,C**

$f(x) = \sum\limits_{k=0}^{n} a_k |x|^k$

$f(x) = a_0|x|^0 + a_1|x| + a_2|x|^2 + \ldots\ldots. + a_n|x|^n = f(|x|)$

f(x) is cont. at x = 0 $\forall$ all is

2k + 1 means all odd a_i's

$f(x) = a_0 + a_2x^2 + a_4x^4 + \ldots\ldots...$

f(x) will be diff. at x = 0

Sol.9 **A,B,D**

f(0) = 0

$f(0^+) = [0^+] = 0$

$f(0^-) = [0^+] = 0$

$f(0^-) = [0^+] = 0$

So f(x) is cont. at x = 0

f(1) = 0

$f(1^+) = -1$ So discont. at x = 1

$\Rightarrow$ Non. diff. at x = 1

Sol.10 B,D

$$f(x) = \lim_{n\to\infty} (\sin x)^{2n}$$

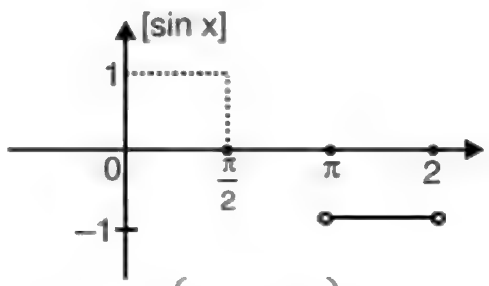
$$f(x) = \begin{cases} 1 ; & x = \pi/2 \\ 0 ; & x < \pi/2 \\ \infty ; & x > \pi/2 \end{cases}$$

$$\left.\begin{array}{l} f\left(\frac{\pi^+}{2}\right) = \infty \\ f\left(\frac{\pi^-}{2}\right) = 0 \end{array}\right\} \text{function is not cont. at } x = \frac{\pi}{2}$$

function is discont at $x = \frac{\pi}{2}$ & infinit number of points.

Sol.11 A,B,D

$$f(x) = \frac{1}{[\sin x]}$$



D_f : [sin x] ≠ 0

$$x \in (2n\pi + \pi, 2n\pi + 2\pi) \cup \left\{2n\pi + \frac{\pi}{2}\right\}$$

cont. when $x \in (2n\pi + \pi, 2n\pi + 2\pi)$

f(x) has the period of 2π

Sol.12 A,B,D

$$f(x) = \sqrt{1-\sqrt{1-x^2}}$$

$D_f : 1 - x^2 \geq 0 \Rightarrow -1 \leq x \leq 1$

RHL (at x = 0) = 0

LHL (at x = 0) = 0 cont. at x = 0

$$\text{RHD} = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{1-\sqrt{1-h^2}} - 0}{h}$$

$$= \lim_{h\to 0} \frac{1}{h} \sqrt{1-\sqrt{1-h^2} \times \frac{1+\sqrt{1-h^2}}{1+\sqrt{1+h^2}}}$$

$$= \lim_{h\to 0} \frac{1}{h} \sqrt{\frac{1-1+h^2}{1+\sqrt{1+h^2}}} = \frac{1}{2}$$

$$\text{LHD} = -\frac{1}{2}$$

Sol.13 A,C

$$f(x) = \lim_{n\to\infty} \frac{1-x^n}{1+x^n} = \begin{cases} 1 ; & x < 1 \\ \infty ; & x > 1 \\ 0 ; & x = 1 \end{cases}$$

$f(1^+) = \infty$

$f(1^-) = 1$

f(x) is a constant in 0 < x < 1

$f'(0^+) \neq f'(0^-)$ not diff. at x = 1

Sol.14 B,C

$$f\left(\frac{1}{4^n}\right) = (\sin e^n)\, e^{-n^2} + \frac{n^2}{1+n^2}$$

put n = ∞

f(0) = [{a finite quantity b/w (–1, 1)} × 0] + 1 = 1

Sol.15 C,D

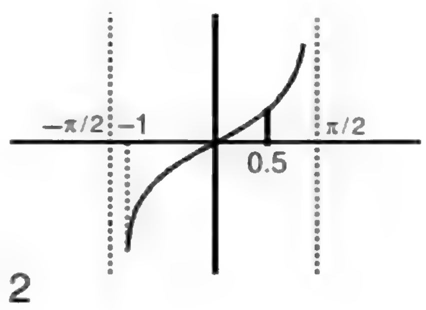
$$f(x) = \frac{x}{2} - 1 \qquad \text{on } [0, \pi]$$

$$f(x) = \frac{x-2}{2} : \frac{1}{f(x)} = \frac{2}{x-2} \qquad 0 \leq \frac{x}{2} \leq \frac{\pi}{2}$$

$$f^{-1}(x) = 2(1 + x) \text{ is cont.} \quad -1 \leq \frac{x}{2} - 1 < \frac{\pi}{2} - 1 \approx 0.5$$

$$\tan f(x) = \tan\left(\frac{x-2}{2}\right)$$

is cont.



$\frac{1}{f(x)}$ will discont at x = 2

Sol.16 A,C

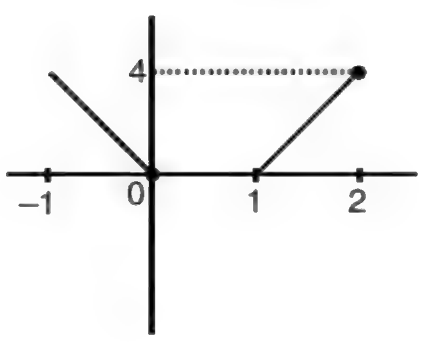
f(x) = |[x]x| $-1 \leq x \leq 2$

$$f(x) = \begin{cases} -x & -1 \leq x < 0 \\ 0 & 0 \leq x < 1 \\ x & 1 \leq x < 2 \\ 4 & x = 2 \end{cases}$$

$\lim_{x\to 0^+} 0 = 0 : \lim_{x\to 0^-} 0 = 0$

cont. at x = 0

Not diff. at x = 2



Sol.17 A,B

$f(x) = 1 + x\,.\,[\cos x] \qquad 0 < x \le \dfrac{\pi}{2}$

$f\left(\dfrac{\pi}{2}\right) = 1 = f\left(\dfrac{\pi}{2}^-\right)$

function is cont. is $\left(0, \dfrac{\pi}{2}\right]$

$f'\left(\dfrac{\pi}{2}^-\right) = \lim\limits_{h\to 0} \dfrac{1-h[\cos(-h)]-1}{-h} = 1$

diff. at $x = \dfrac{\pi}{2}$

Sol.18 B,D

$f(x) = (\sin^{-1} x)^2\,.\,\cos\left(\dfrac{1}{x}\right)$ if $x \ne 0$

$= 0$ if $x = 0$

$\text{LHL} = \text{RHL} = \lim\limits_{x\to 0} (\sin^{-1} x)^2 \cos\left(\dfrac{1}{x}\right)$

$= 0 \times$ [a finite quantity b/w $(-1, 1)$]

$= 0$

$f'(0^+) = \lim\limits_{h\to 0} \dfrac{(\sin^{-1} h)^2 \cos\left(\dfrac{1}{h}\right)}{h}$

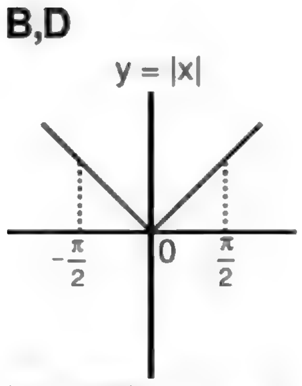
$= \lim\limits_{h\to 0} \left(\dfrac{\sin^{-1} h}{h}\right) (\sin^{-1} h)\cos(1/h)$

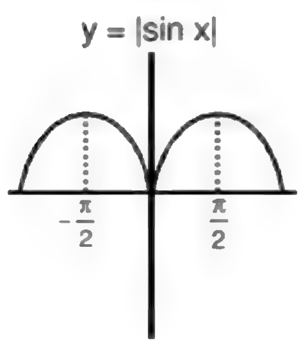
$= 1 \times (0) \times$ (a finite quantity)

$= 0$

$f'(0^-) = 0$

Sol.19 B,D





Not diff. at $x = 0$

Sol.20 A,B,C

$f(x) = 3(2x + 3)^{2/3} + 2x + 3$

$f\left(\dfrac{-3}{2}\right) = 0 - 3 + 3 = 0$

cont. every where

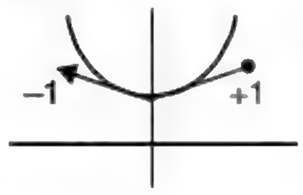
$f'(x) = -2\,(2x + 3)^{-1/3} + 2$

$= -\dfrac{2}{(2x+3)^{1/3}} + 2$

at $x = -\dfrac{3}{2}$; $f'(x)$ is not defined

Sol.21 B,D

$f(x) = 2 + |\sin^{-1} x|$



function is continuous everywhere in its domain but $f(x)$ is not diff. at $x = 0$

Sol.22 A,B,D

$f(x) = x^2 \sin\left(\dfrac{1}{x}\right) \qquad , \; x \ne 0$

$= 0 \qquad , \; x = 0$

cont. at $x = 0$

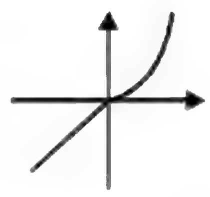
$f(0^+) = f(0^-) = f(0) = 0$

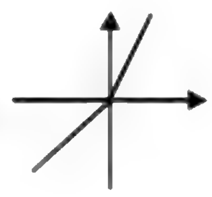
$f'(0^+) = \lim\limits_{h\to 0} \dfrac{h^2 \sin 1/h}{h} = \lim\limits_{h\to 0} h \sin\dfrac{1}{h} = 0$

$f'(0^-) = 0$

Diff. at $x = 0$

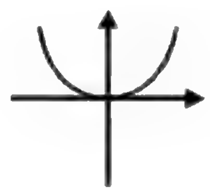
Sol.23 A,B,D

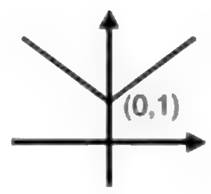
(A) 

(B) 

(C) $h(x) = x^2 \quad x \ge 0$

$= -x^2 \; x < 0$



(D) 

Sol.24 A,B,D

$\sin^{-1} x + |y| = 2y$

$\sin^{-1} x = 2y - y$

$y = \sin^{-1} x$

y is defined for $-1 \le x \le 1$

EXERCISE – III HINTS & SOLUTIONS

Sol.1 $f(x) = \dfrac{3x^3 + ax + a + 3}{x^2 + x - 2}$

$$\lim_{x\to-2} \frac{3x^2+ax+a+3}{x^2+x-2} = f(-2)$$

$\Rightarrow$ format is $\dfrac{15-a}{0}$, for existence of limit $N^r = 0$

$\Rightarrow \quad a = 15$

$$\lim_{x\to-2} \frac{3x^2+15x+18}{x^2+x-2} = f(-x)$$

R.H.S. $\displaystyle\lim_{h\to0} \frac{3(-2+h)^2+15(-2+h)}{(-2+h)^2+(-2+h)-2} = f(-2)$

$$\lim_{h\to0} \frac{12+3h^2-12h-30+15h+18}{h^2-3h} = f(-2)$$

$$\lim_{h\to0} \frac{3h^2+3h}{h^2-3h} = f(-2)$$

$\boxed{f(-2) = -1}$

Sol.2 $f(x) = \begin{cases} |ax+3| & x \le -1 \\ |3x+a| & -1 < x \le 0 \\ \dfrac{b\sin 2x}{x} - 2b & a < x < \pi \\ \cos^2 x - 3 & x \ge \pi \end{cases}$

$f(0^-) = f(0)$

$f(x^-) = \dfrac{-b\sin 2h}{-h} - 2b = |a|$

$+ 2b - 2b = |a| \quad \Rightarrow \quad x = 0$

$f(\pi^-) = f(\pi)$

$$\lim_{h\to0} \frac{b\sin(\pi-h)}{(\pi-h)} - 2b = -2$$

$-2b = -2 \qquad \Rightarrow \qquad b = 1$

So, $\boxed{a=0}$ & $\boxed{b=1}$

Sol.3 $f(x) = \begin{cases} \dfrac{\ell n \cos x}{\sqrt[4]{1+x^2}-1} & x > 0 \\ \dfrac{e^{\sin 4x}-1}{\ell n(1+\tan 2x)} & x < 0 \end{cases}$

$$f(0^+) = \lim_{h\to0} \frac{\ln\cos h}{(1+h^2)^{1/4}-1}$$

$$= \lim_{h\to0} \frac{\ln(1+\cos h-1)\}(\cos h-1)}{(\cos h-1)\{1+h^2)^{1/4}-1\}}$$

$$= \lim_{h\to0} \frac{(\cos h-1)}{(1+h^2)^{1/4}-1} \Rightarrow \lim_{h\to0} \frac{\dfrac{\cos h-1}{h^2}\times h^2}{\left(x+\dfrac{h^2}{4}-1\right)}$$

$$\Rightarrow \qquad -\frac{1}{2}\times 4 = -2$$

$$f(0^-) = \lim_{h\to0} \frac{e^{\sin(-4h)}}{\ell n(1-\tan 2h)}$$

$$= \lim_{h\to0} \frac{\{e^{-\sin 4h}-1\}(-\tan 2h)}{\ell n\{1+(-\tan 2h)\}(-\tan 2h)} = \lim_{h\to0} \frac{e^{-\sin 4h}-1}{-\dfrac{\tan 2h}{2h}\times 2h}$$

$$= -\lim_{h\to0} \frac{e^{-\sin 4h}-1}{2h} = \lim_{h\to0} \frac{e^{\sin 4h}-1}{2h(e^{\sin 4h})}$$

$$= \lim_{h\to0} \frac{(e^{\sin 4h}-1)\sin 4h}{\sin 4h \times 2he^{\sin 4h}} = \lim_{h\to0} \frac{2}{e^{\sin 4h}} = 2$$

$f(0) \to = -2$ & $f(0^+) = e$

Hence it is discontinuity of non removable type.

Sol.4 $f(x) = x^3 - 3x^3 - 4x + 12$ &

$h(x) = \begin{cases} \dfrac{f(x)}{x-3} & , x \ne 3 \\ k & , x = 3 \end{cases}$

$h(x) = \begin{cases} \dfrac{x^3-3x^2-4x+12}{x-3} & , x \ne 3 \\ k & , x = 3 \end{cases}$

$f(x) = f(3^+) = f(3)$

$\lim_{x\to 3} f(x) = f(3)$

$$\lim_{x\to 3} \frac{a^3 - 3x^2 - 4x + 12}{x - 3} = k$$

$\boxed{5 = k}$

Zero of f(x) $\Rightarrow$ $x = 3, 2, -2$

$$h(x) = \begin{cases} \dfrac{x^3 - 3x^2 - 4x + 12}{x - 3} & , x \neq 3 \\ k & , x = 3 \end{cases}$$

$h(x) = x^2 - 4$

& checking at $x = -3$

$h(x) = 5 = k$

hence then.

Sol.5

$$y_n(x) = x^2 + \frac{x^2}{1 + x^2} + \frac{x^2}{(1 + x^2)^2} + ... + \frac{x^2}{(1 + x^2)^{n-1}}$$

$$y_n(x) = \frac{x^2(1 + x^2)\left(1 - \dfrac{1}{(1 + x^2)^2}\right)}{x^2}$$

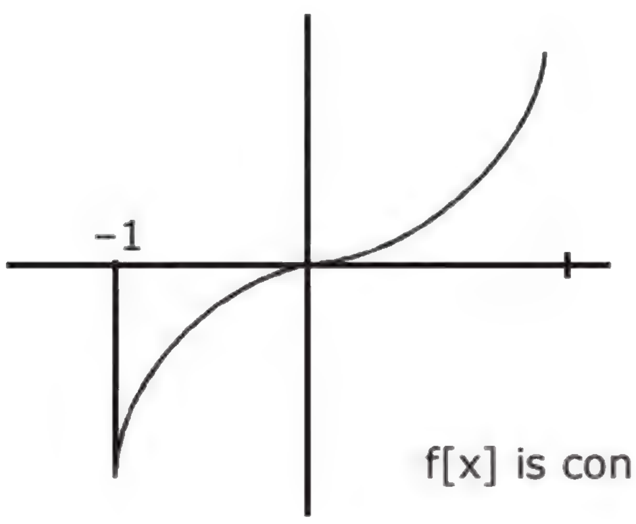
$$\lim_{x\to 0} y_n(x) = \lim_{x\to 0} (1 + x^2)\left[1 - \frac{1}{(1 + x^2)^n}\right] = 0 = y_n(0)$$

Sol.6 $f(x) = x - |x - x^2|, x \in [-1, 1]$

$f(x) = x - |x - x^2| \quad -1 \le x \le 1$

$f(x) = x(2 - x) \quad -1 \le x < 0$

$= x^2 \quad 0 \le x \le 1$



f[x] is con is [-1,]

f(x) is cont is [-1, 1]

Sol.7
$$f(x) = \begin{cases} \dfrac{1 - \sin \pi x}{1 + \cos 2\pi x} & x < \dfrac{1}{2} \\ P & x = \dfrac{1}{2} \\ \dfrac{\sqrt{2x - 1}}{\sqrt{4 + \sqrt{x - 1} - 2}} & x > \dfrac{1}{2} \end{cases}$$

$$f\left(\frac{1^-}{2}\right) = \lim_{h\to 0} \frac{1 - \sin\pi\left(\frac{1}{2} - h\right)}{1 + \cos 2\pi\left(\frac{1}{2} - h\right)}$$

$$= \lim_{h\to 0} \frac{1 - \cos \pi h}{1 - \cos 2\pi h} \Rightarrow \lim_{h\to 0} \frac{2\sin^2\left(\frac{\pi h}{2}\right)}{2\sin^2 \pi h}$$

$$f\left(\frac{1^-}{2}\right) = \lim_{h\to 0} \frac{\sin^2\left(\frac{\pi h}{2}\right)}{\frac{\sin^2 \pi h}{\pi^2 b^2} \times \pi^2 b^2} \times \frac{\pi^2 h^2}{4} \times \frac{4}{\pi^2 h^2}$$

$$= \frac{1}{4}$$

$$f\left(\frac{1^+}{2}\right) = \lim_{h\to 0} \frac{\sqrt{2\left(\frac{1}{2} + h\right) - 1}}{\sqrt{4 + \sqrt{2\left(\frac{1}{2} + h\right) - 1} - 2}}$$

$$= \lim_{h\to 0} \frac{\sqrt{2h}}{\sqrt{4\sqrt{2h} - 2}}$$

$$= \lim_{h\to 0} \frac{\sqrt{2h}}{\sqrt{2 + \sqrt{2h}}} = 0$$

There is no value of 'P' because f(x) is not cointinuous.

Sol.8 $g(x) = \sqrt{6 - 2x}$

$h(x) = 2x^2 - 3x + a$

(a) $h(g(x)) \Rightarrow h\left(\sqrt{6 - 2\times 2}\right) \Rightarrow h\ \sqrt{2}$

$\Rightarrow 2 \times 2 - 3\sqrt{2} + a$

$\Rightarrow 4 - 3\sqrt{2} + a$

(b) $f(x) = \begin{cases} g(x) & ; \quad X \le 1 \\ h(x) & ; \quad x > 1 \end{cases}$

$$f(x) = \begin{cases} \sqrt{6-2x} & ; \quad X \le 1 \\ 2x^2 - 3x + a & ; \quad x > 1 \end{cases}$$

$$f(I^{-1}) = \lim_{h \to 0} \sqrt{6-2+2h} = h$$

$$f(I^{+}) = \lim_{h \to 0} 2(1+h^{-})^2 - (1+h) + 2$$

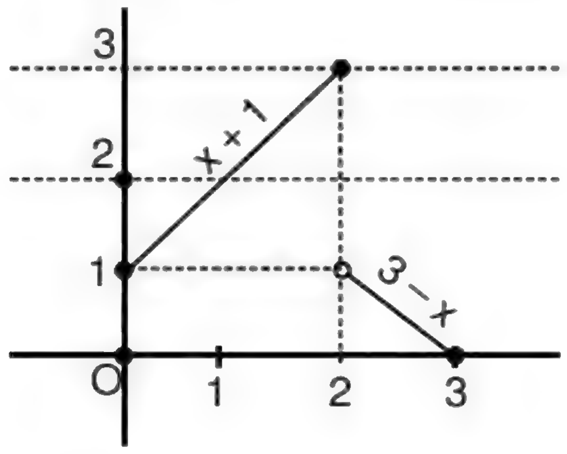
$$= -1 + a$$

If f(x) is cont. a – 1 = 2

$$\boxed{a = 3}$$

Sol.9 $f(x) = \begin{cases} 1+x & ; \quad 0 \le x \le 2 \\ 3-x & ; \quad 2 < x \le 3 \end{cases}$

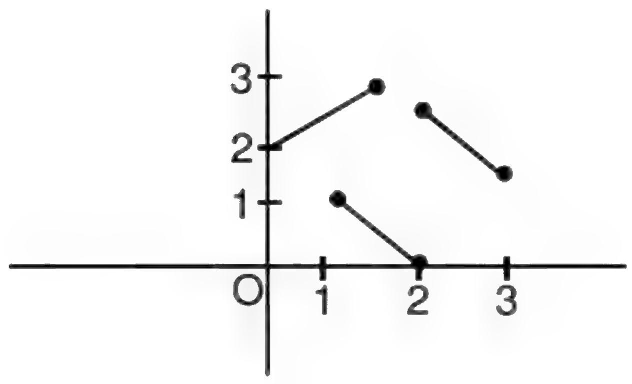
$$g(x) = f(f(x)) = \begin{cases} 1+f(x) & ; \quad 0 \le f(x) \le 2 \\ 3-f(x) & ; \quad 2 < f(x) \le 3 \end{cases}$$



$$0 \le f(x) \le 2 \begin{cases} \rightarrow 1 + x + 1 \ ; \ 0 \le x \le 1 \\ \rightarrow 1 + 3 - x \ ; \ 2 < x \le 3 \end{cases}$$

$2 < f(x) \le 3 \longrightarrow 3 - (x+1) \ 1 \le x \le 2$

$$g(x) = \begin{cases} x+2 & ; \quad 0 \le x \le 1 \\ 2-x & ; \quad 1 < x \le 2 \\ 4-x & ; \quad 2 < x \le 3 \end{cases}$$



Pt. of disc (s) are 1 & 2.

Sol.10 $f(x) = \begin{cases} \dfrac{e^{\{(x+2)\ln 4\}^{\frac{[x+1]}{4}}} - 16}{4^x - 16} & ; \quad x < 2 \\ A\dfrac{1-\cos(x-2)}{(x-2)\tan(x-2)} & ; \quad x > 2 \end{cases}$

$$f(2^-) = \lim_{x \to 2^-} \frac{(4^{(x+2)})^{\frac{[x+1]}{4}} - 16}{4^{2-h} - 16}$$

$$= \lim_{h \to 0} \frac{(4^{(4-h)})^{\frac{2}{4}} - 16}{4^{2-h} - 16} \Rightarrow \lim_{h \to 0} \frac{4^{2-\frac{h}{2}} - 16}{4^{2-h} - 16}$$

$$\lim_{h \to 0} \frac{4^{2-\frac{h}{2}}\left\{1 - 4^{2-2+\frac{h}{2}}\right\}}{4^{2-h}\{1 - 4^{2-2+h}\}}$$

$$\lim_{h \to 0} 4^{\frac{h}{2}} \left\{ \frac{1 - 4^{\frac{4}{2}}}{1 - 4^h} \right\}$$

$$\lim_{h \to 0} 4^{\frac{h}{2}} \left\{ \frac{\frac{4^h - 1}{1} \times 4}{\frac{4^h - 1}{4} \times 4} \right\} = \frac{L}{2}$$

$$f(2^+) = \lim_{x \to 2^+} A \frac{1-\cos(x-2)}{(x-2)\tan(x-2)}$$

$$= \lim_{h \to 0} A \frac{(1-\cos h)}{\frac{h(\tan h)}{h} \times h} = \frac{A}{2}$$

If f(x) is cont. then, A = 1 & $f(2) = \frac{1}{2}$

Sol.11 $f(x) = \begin{cases} \left(\dfrac{6}{5}\right)^{\frac{\tan 6x}{\tan 5x}} & 0 < x < \dfrac{\pi}{2} \\ b+2 & x = \dfrac{\pi}{2} \\ (1+|\cos x|)^{\frac{a|\tan x|}{b}} & \dfrac{\pi}{2} < x < \pi \end{cases}$

$$f\left(\frac{\pi^-}{2}\right) = \lim_{h\to 0}\left(\frac{6}{5}\right)^{\frac{\tan 6\left(\frac{\pi}{2}-h\right)}{\tan 5\left(\frac{\pi}{2}-h\right)}}$$

$$= \lim_{h\to 0}\left(\frac{6}{5}\right)^{\frac{\tan(3x-6h)}{\tan\left(\frac{5\pi}{2}-5h\right)}} = \lim_{h\to 0}\left(\frac{6}{5}\right)^{\frac{-\tan 6h}{\cot 5h}} = 1$$

$$f\left(\frac{\pi^+}{2}\right) = \lim_{x\to\frac{\pi^+}{2}} (1+\cos x)^{\frac{a(\tan x)}{b}}$$

$$\ell = \lim_{h\to 0} -\frac{a\cot h}{b}(-\sin h)$$

$$\ell = \lim_{h\to 0} + \frac{a\cos h}{b}$$

$$\ell = \frac{a}{b} \Rightarrow e^{a/b}$$

$$f\left(\frac{\pi}{2}\right) = b + 2 \Rightarrow \boxed{b=1} \neq e^a = 1 \Rightarrow \boxed{a=0}$$

Sol.12 $$f(x) = \begin{cases} \dfrac{1-\sin^3 x}{3\cos^2 x} & ; \ x < \dfrac{\pi}{2} \\ a & ; \ x = \dfrac{\pi}{2} \\ \dfrac{b(1-\sin x)}{(\pi-2x)^2} & ; \ x > \dfrac{\pi}{2} \end{cases}$$

$$f\left(\frac{\pi^-}{2}\right) = \lim_{x\to\frac{\pi^-}{2}} \frac{1-\sin^3 x}{3\cos^2 x}$$

$$= \lim_{h\to 0} \frac{1-\cos^3 h}{3\sin^2 h}$$

$$\Rightarrow \lim_{h\to 0} \frac{(1-\cos h)(1+\cos^2 h+\cos h)}{3\dfrac{\sin^2 h}{h^2}\times h^2}$$

$$= \frac{1}{6}\times(1+1+1) = \frac{1}{2}$$

$f\left(\frac{\pi}{2}\right)$ = a. since f(x) is cont at $x = \frac{\pi}{2}$.

$$\text{Hence} = f\left(\frac{\pi^-}{2}\right) = f\left(\frac{\pi}{2}\right)$$

So, $\boxed{a = \frac{1}{2}}$

$$\text{Now, } f\left(\frac{\pi^+}{2}\right) = \lim_{h\to 0}\frac{b(1-\cos h)}{4h^2} = \frac{b}{2\times 4}$$

$$\Rightarrow \boxed{b=4}$$

Sol.13 $$f(x) = \begin{cases} \dfrac{\sin(a+1)x+\sin x}{x} & ; \ x < 0 \\ c & ; \ x = 0 \\ \dfrac{(x+bx^2)^{\frac{1}{2}} - x^{\frac{1}{2}}}{bx^{3/2}} & ; \ x > 0 \end{cases}$$

$$f(0^-) = \lim_{h\to 0}\frac{\sin(a+1)(-h)-\sin h}{-h}$$

$$= \lim_{h\to 0}\frac{[\sin(a+1)h](a+1)}{h(a+1)} + \lim_{h\to 0}\frac{\sin h}{h}$$

$$= a + 1 + 1 = \boxed{a+2}$$

f(0) = c,

$$f(0^+) = \lim_{h\to 0}\frac{(h+bh^2)^{\frac{1}{2}} - h^{\frac{1}{2}}}{bh^{\frac{3}{2}}}$$

$$= \lim_{h\to 0}\frac{h+bh^2-h}{bh^{3/2}\{(h+bh^2)^{1/2}+h^{1/2}\}}$$

$$\Rightarrow \lim_{h\to 0}\frac{h^{1/2}}{\{h^{1/2}(1+bh)^{1/2}+1\}} = 1/2$$

Since 'f' is cont at x = 0, so,

$f(0^+) = f(0^-) = f(0)$

$$\frac{1}{2} = a + 2 = c \Rightarrow \boxed{C = \frac{1}{2}}$$

Sol.14 $$f(x) = \frac{\sin 3x + A\sin 2x + B\sin x}{x^5}$$

$$= \lim_{x\to 0} \frac{\left\{3x - \frac{(3x)^3}{3} + \frac{(3x)^5}{5} + A\right\} + A\left\{2x - \frac{(2x)^3}{3} + \frac{(2x)^5}{5}\right\} + B\left\{x - \frac{x^3}{3} + \frac{x^5}{5}\right\}}{x^5}$$

$$\lim_{x\to 0} \frac{(3+2A+B)x}{x^5} - \frac{x^3\left(9 + \frac{8A}{3} + \frac{B}{3}\right)}{x^5}$$

$$+ \frac{x^5}{x^5}\left(\frac{3^5}{5} + \frac{32A}{5} + \frac{B}{5}\right)$$

For existence of limit,

$$3 + 2A + B = 0 \ \& \ 9 + \frac{8A}{3} + \frac{B}{3} = 0$$

$$2A + B + 3 = 0 \ \& \ 27 + 8A + B = 0$$

$$6A + 24 = 0 \quad \Rightarrow \quad \boxed{A = -4}$$

$$\boxed{B = 5}$$

$$f(0) = \frac{243}{5} - \frac{128}{5} + 1$$

$$f(0) = \frac{115}{5} + 1 \ \Rightarrow \quad \boxed{24}$$

Sol.15

$$f(x) = \begin{cases} \dfrac{\left\{\frac{\pi/2}{2} - \sin^{-1}(1-\{x\}^2)\right\}\sin^{-1}(1-\{X\})}{\sqrt{2}(\{x\}-\{x\}^3)} & , \ x \neq 0 \\ \dfrac{\pi}{2} & , \ x = 0 \end{cases}$$

$$g(x) = \begin{cases} f(x) & , \ x \geq 0 \\ 2\sqrt{2}f(x) & , \ x < 0 \end{cases}$$

$$f(0^-) = \lim_{h\to 0} \frac{\left\{\frac{\pi}{2} - \sin^{1}(1-(1-h)^2)\right\}\sin^{-1}(1-1-h)}{\sqrt{2}\{1-h-(1-h)^3\}}$$

$$= \lim_{h\to 0} \frac{\left\{\frac{\pi}{2} - \sin^{-1}(1-1-h^2+2h)\right\}\sin^{-1}h}{\sqrt{2}(1-h)(1-h^2-h-1)}$$

$$= \lim_{h\to 0} \frac{\left\{\frac{\pi}{2} - \sin^{-1}(2h-h^2)\right\}\sin^{-1}h}{\sqrt{2}(1-h)(1-h)(h+1)}$$

$$= -\lim_{h\to 0} \frac{\cos^{-1}(2h-h^2)}{\sqrt{2}(1-h^2)} = \frac{-\pi}{2\sqrt{2}}$$

$$f(0^+) = \lim_{h\to 0} \frac{\left\{\frac{\pi}{2} - \sin^{-1}(1-h^2)\right\}\left\{\sin^{-1}(1-h)\right\}}{\sqrt{2}(h-h^3)}$$

$$= \lim_{h\to 0} \left\{\frac{\cos^{-1}(1-h^2).\sin^{-1}(1-h)}{\sqrt{2}\,h(1+h)(1-h)}\right\}$$

$$= \lim_{h\to 0} \frac{\sin^{-1}(1-h)}{\sqrt{1-h^2}} = \frac{\pi}{2}$$

$$g(x) = \begin{cases} \dfrac{\left\{\frac{\pi}{2} - \sin^{-1}(1-\{x\}^2)\right\}\sin^{-1}(1-\{x\})}{\sqrt{2}(\{x\}-\{x\}^3)} & x \geq 0 \end{cases}$$

Sol.16 $f(x) = \begin{cases} |4x-5|[x] & x > 1 \\ [\cos \pi x] & x \leq 1 \end{cases}$

Defining the given fn as follows :

$$f(x) = \begin{cases} 1 & x = 0 \\ 0 & \left(0, \frac{1}{2}\right) \\ -1 & \left(\frac{1}{2}, 1\right] \\ & \left(1, \frac{5}{4}\right) \\ & x = 2 \end{cases}$$

Sol.17 $f(x) = x + \{-x\} + [x]$

$= x + (-x - [x]) + [x]$

$f(x) = [x] - [-x]$

$$f(x) = \begin{cases} -3 & -2 \leq x < -1 \\ -1 & -1 \leq x < 0 \\ 1 & 0 \leq x < 1 \\ 3 & 1 \leq x < 2 \\ 4 & x = 2 \end{cases}$$

f(x) discont. at every intiger value is [–2, 2]

Sol.18 $f(x) = \begin{cases} 3x - b & ; \ x \leq 1 \\ 3x & ; \ 1 < x < 2 \\ 5x^2 - a & ; \ x \geq 2 \end{cases}$

at x = 1

LHL = a – b ; RHL = 3
LHL = RHL
$\Rightarrow$ a – b = 3
Locus x – y = 3 .....(i)
at x = 2
LHL = 6 ; RHL = 4b – a
4b – a = 6
or 4y – x = 6 .....(ii)
from (i) & (ii)
y = 3
But $y \neq 3$ because discont. at x= 2

Sol.19 $f(x) = \lim_{n \to \infty} \frac{ax^2 + bx + c + (e^x)^n}{1 + c(e^x)^n}$

$$f(x) = \begin{cases} 1/c & ; \; e^x > 1 \Rightarrow x > 0 \\ \frac{ax^2 + bx + c + 1}{1 + c} = 1 & ; \; e^x = 1 \Rightarrow x > 0 \\ ax^2 + bx + c & ; \; e^x < 1 \Rightarrow x < 0 \end{cases}$$

For cont.
$f(0) = f(0^+) = f(0^-)$

$1 = \frac{1}{c} \Rightarrow c = 1$

$a, b \in R$

Sol.20 $g(x) = \lim_{n \to \infty} \frac{x^n f(x) + h(x) + 1}{2x^n + 3x + 3}$

$$g(x) = \begin{cases} \frac{f(x)}{2} & ; \; x > 1 \\ \frac{f(1) + h(1) + 1}{8} & ; \; x = 1 \\ \frac{h(x) + 1}{3x + 3} & ; \; x < 1 \end{cases}$$

$$g(x) = \lim_{x \to 1} \frac{\text{Sin}^2(\pi 2^x)}{\ln(\text{Sec}(\pi - 2^x))}$$

$$= \lim_{x \to 1} \frac{-\text{Sin}^2(\pi 2^x)}{\ln\frac{(1 + \cos \pi 2^x - 1)}{\cos \pi 2^x - 1}} \times \frac{1}{\cos \pi . 2^x - 1}$$

$$= \lim_{x \to 1} \frac{\text{Sin}^2(2\pi - \pi 2^x)}{\left(\frac{1 - \cos \pi 2^x - 1}{(\pi . 2^x)}\right) \times (\pi 2^x)}$$

g(1) = 2

$g(1) = \frac{f(1) + h(1) + 1}{8} = 2$

$\Rightarrow$ f(1) +h(1) =15 -----(i)
$g(1^+) = (1^-)$

$\frac{f(1)}{2} = \frac{h(1) + 1}{6} \Rightarrow$ 3fu) = hu) + 1 ----- (ii)

from (ii) & (i)
h (1) = 11 and f(1) = 4
4g(1) + 2f(1) – h(1)
4g(1) + 2(f(1) + h(1)) – 3h (1)
= 8 + 2 (15) – 3 $\times$ 11 = 5

Sol.21 $f(x) = \frac{2 + \cos x}{x^3 \sin x} - \frac{3}{x^4}$

$$= \frac{2x + x\cos x - 3\sin x}{x^4 \sin x}$$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{2x + x\cos x - 3\sin x}{x^4 \sin x}$$

$$= \lim_{x \to 0} \frac{2x + x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - - - -\right) - 3\left(1 - \frac{x^3}{3!} + \frac{x5}{5!} - - - -\right)}{x^5}$$

$$\lim_{x \to 0} \frac{\left(\frac{4}{4!} - \frac{3}{5!}\right)x^5 + - - - -}{x^5}$$

$$f(0) = \frac{1}{60}$$

Sol.22 $f(x) \; \frac{a^{\tan x}(a^{\sin x - \tan x} - 1)}{-(\sin x - \tan x)}$

$$= \lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{a^{\tan x}(a^{\sin x - \tan x} - 1)}{-(\sin x - \tan x)}$$

$$= -\ln a = \ln\left(\frac{1}{a}\right) \text{-----(i)}$$

$$\text{LHL} = \lim_{x \to 0} f(x) \lim_{x \to 0} \frac{\ln(1 + x^2 + x^4)}{\sec x - \cos x}$$

$$= \lim_{x \to 0} \frac{\cos x}{(1-\cos x)(1+\cos x)} \times$$

$$\frac{\ln(1+x^2+x^4)}{(x^2+x^4)} \times (x^2+x^4)$$

$$\lim_{x \to 0} \frac{x^2(1+x^2)}{(1-\cos x)(1+\cos x)} = 1$$

LHL = RHL

$$\ln\left(\frac{1}{a}\right) = 1 \Rightarrow a = \frac{1}{e}$$

$$g(x) = \ln\left(2 - \frac{x}{a}\right)\cot(x-a)$$

$$\lim_{x \to \frac{1}{e}} \ln\left(2 - \frac{x}{a}\right)\cot\left(x - \frac{1}{e}\right)$$

$$\lim_{x \to \frac{1}{e}} \frac{\ln(1+1-xe)}{(1-xe)} \times \frac{(1-xe)}{\left(x - \frac{1}{e}\right)}$$

$$\lim_{x \to \frac{1}{e}} g(n) = -e$$

"If g(n) is continous

then $g\left(\frac{1}{e}\right) = -e$

Sol.23 $f(x+y)= f(x) + f(y)$

put $x = 0$, $y = 0$

$\Rightarrow f(0) = 0$ ------(i)

Let $x = a$

$f(a^+) = \lim_{x \to a^+} f(x)$

$= \lim_{x \to 0} f(a+h)$

$= \lim_{x \to 0} f(a) + f(n) = f(a)$ -----(ii)

$f(a^-) = \lim_{x \to a^-} f(x)$

$= \lim_{h \to 0} f(a-h)$

$= \lim_{h \to 0} f(a) + f(-h)$

$f(a^-) = f(a) + f(-h)$---- (iii)

from (ii) & (iii)

$f(a^+) = f(a^-) = f(a)$

so $f(x)$ is cont. at $x = a$

$\Rightarrow f(x)$ is cont. $\forall$ all $x \in R$

Sol.24 $f(x) = \sum_{r=1}^{n} \tan\left(\frac{x}{2^r}\right) \sec\left(\frac{x}{2^{r-1}}\right)$; $r, n \in N$

Let $T_r = \tan\left(\frac{x}{2^r}\right) \sec\left(\frac{x}{2^{r-1}}\right)$ $\theta = \frac{x}{2^r}$

$= \tan\theta \sec 2\theta$

$$T_r = \frac{\sin\theta}{\cos\theta\cos 2\theta} = \frac{\sin(2\theta-\theta)}{\cos\theta\cos 2\theta}$$

$= \tan 2\theta - \tan\theta$

$$T_r = \tan\frac{x}{2^{r-1}} - \tan\frac{x}{2^r}$$

$$T_1 = \tan\frac{x}{2^0} - \tan\frac{x}{2}$$

$$T_2 = \tan\frac{x}{2} - \tan\frac{x}{2^2}$$

$\vdots$

$$T_n = \tan\frac{x}{2^{n-1}} - \tan\frac{x}{2^n}$$

$f(x) = $ sum of T_r

$$f(x) = \tan x - \tan\frac{x}{2^n}$$

$$f(x) + \tan\frac{x}{2^n} = \tan x$$

$$g(x) = \lim_{n \to \infty} \frac{\ell n \tan x - (\tan x)^n \left[\sin\left(\tan\frac{x}{2}\right)\right]}{1+(\tan x)^n}, x \neq \frac{\pi}{4}$$

$g(x) = -\left[\sin\left(\tan\frac{x}{2}\right)\right]$ $\quad \tan x > 1$

$\Rightarrow x > \frac{\pi}{4}$

$= \ell n(\tan x)$ $\quad \tan x < 1$

$\Rightarrow x < \frac{\pi}{4}$

LHL $\left(\text{at } x = \frac{\pi}{4}\right) = 0$

RHL $\left(\text{at } x = \frac{\pi}{4}\right) = -\left[\sin\left(\tan\frac{\pi}{8}\right)\right]$

$= 0$

$\Rightarrow \quad k = 0$

. . . . . $\ell n(\tan x) \quad$ if $0 < x < \frac{\pi}{4}$

$0 \quad$ if $\frac{\pi}{4} \le x < \frac{\pi}{2}$

g(x) is cont. everywhere is $\left(0, \frac{\pi}{2}\right)$

Sol.25 $g(x) = k(x + 1)$

$h(x) = \frac{(x+1)(x^2-2x-1)}{k(x+1)}$

$\lim\limits_{x\to -1} h(x) = \frac{1}{2} \Rightarrow k = 4$

$g(x) = 4(x + 1)$

$h(0) = -1/4$

$f(0) = -1;\ g(0) = 4$

$\lim\limits_{x\to 0} [3h(x) + f(x) - 2g(x)] = \frac{-39}{4}$

Sol.26 (a) Let $f(x) = g(x) - x$

$f(a) = g(a) - a \le 0$

$f(b) = g(b) - b \ge 0$

$\exists \quad c \in [a, b] \Rightarrow f(c) = 0$

$\Rightarrow g(c) = c$

(b) Let $g(x) = f(x) - f\left(x + \frac{1}{2}\right)$

WWTPT $g(c) = 0 \quad \forall\ c \in \left[0, \frac{1}{2}\right]$

$x \in \left[0, \frac{1}{2}\right] \Rightarrow g(x)$ is cont.

$g(0) = f(0) - f\left(\frac{1}{2}\right)$

$g\left(\frac{1}{2}\right) = f\left(\frac{1}{2}\right) - f(1) = f\left(\frac{1}{2}\right) - f(1)$

$= -\left[f(0) - f\left(\frac{1}{2}\right)\right]$

so $g(0)$ & $g\left(\frac{1}{2}\right)$ are opposite signs.

so

$\exists\ c \in \left[0, \frac{1}{2}\right] \Rightarrow g(c) = 0$

$\Rightarrow f(c) = f\left(c + \frac{1}{2}\right)$

Sol.27

$g(0^-) = \lim\limits_{x\to 0^-} \frac{1 - a^x + a^x \ell n\, a^x}{a^x . x^2}$ Let $a^x = t$

$= \ell n^2 a \lim\limits_{t\to 1} \frac{1 - t + t\,\ell n\, t}{\ell n^2 t}$

$= \frac{1}{2} \ell n^2 a$

$g(0^+) = \lim\limits_{x\to 0^+} \frac{(2a)^x - \ell n(2a)^x - 1}{\ell n^2 t}$

$= \ell n^2 2a \lim\limits_{t\to 1} \frac{t - \ell n\, t - 1}{\ell n^2 t} = \frac{1}{2} \ell n^2 2a$

$g(0^+) = g(0^-)$

$\frac{1}{2} \ell n^2 2a = \frac{1}{2} \ell n^2 a \Rightarrow a = \frac{1}{\sqrt{2}}$

$\Rightarrow g(0) = \frac{1}{2} \ell n^2 \frac{1}{\sqrt{2}} = \frac{1}{8} (\ell n\, 2)^2$

EXERCISE – IV HINTS & SOLUTIONS

Sol.1 Let $f_1(x) = \sin x$ & $f_2(x) = \sin |x|$
$f_1(x)$ is continuous & differentiable always.
$f_2(x)$ is continuous but not differentiable at $x = 0$.
so $f_1(x) + f_2(x) = f(x)$ is continious but not differentiable at $x = 0$.
(using fundamental theorems)

Sol.2 $f(x) = \begin{cases} -3x+3 & x<0 \\ -x+3 & x \in [0,1) \\ x+1 & x \in [1,2) \\ 3x-3 & x>2 \end{cases}$

so continuous but non differentiable at $x = 0, 1, 2$.

Sol.3 for differentiability

$$f'(0^+) = \lim_{h\to 0} \frac{h^n \sin 1/h}{h}$$

so only if $n \in (0, 1]$, function is non differentiable.

Sol.4 $f'(0^+) = \lim_{h\to 0} \dfrac{f(0+h)-f(0)}{h}$

$$= \lim_{h\to 0} \frac{1+\sin h-1}{h} = 1$$

$$f'(0^-) = \lim_{h\to 0} \frac{f(0-h)-f(0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0$$

so continuous but not differentiable at $x = 0$
(check at $x = \pi/2$)

Sol.5 $f'(0^+) = \lim_{h\to 0} \dfrac{f(0+h)-f(0)}{h}$

$$= \lim_{h\to 0} \frac{h.\tan^{-1}(1/h)-0}{h}$$

$$= \frac{\pi}{2}$$

$$f'(0^-) = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0} -\tan^{-1}(1/h)$$

$= -\pi/2$.
so continuous but non differentiable at $x = 0$.

Sol.6 LHS = $\lim_{x\to 0} \int \dfrac{f(x)-f(0)}{x}$

$$+ \frac{1}{2}\left\{\frac{f(x/2)-f(0)}{x/2}\right\} +\ldots\ldots + \frac{1}{k}\left\{\frac{f(x/k)-f(0)}{x/k}\right\}$$

$$= f'(0) + \frac{f'(0)}{2} +\ldots\ldots+ \frac{f'(0)}{k} = 1 + \frac{1}{2} +\ldots\ldots+ \frac{1}{k}$$

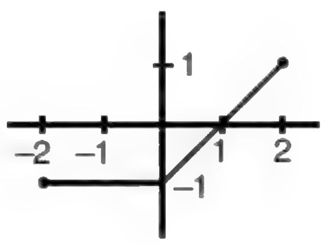
Sol.7 $f'(0^+) = 0$
$f'(0^-) = 1$
so non differentiable but continuous at $x = 0$.

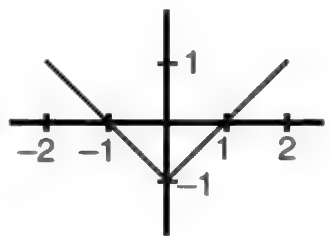
Sol.8 $f'(1^+) = 3$ & $f'(1^-) = -1$

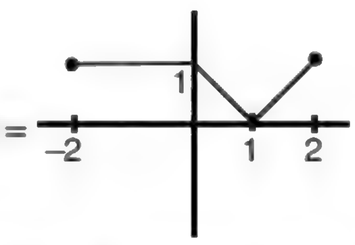
Sol.9 If f(x) is continuous then
$f(1^-) = f(1^+) \Rightarrow a(1) - b = -1$
$\Rightarrow \quad a - b = -1$
If f(x) is differentiable then
$f'(1^+) = f'(1^-)$

$$\Rightarrow \quad \frac{1}{x^2} = 2ax\ ;\ \text{at } x = 1 \Rightarrow a = 1/2$$

so $\quad b = 3/2$

Sol.10 f(x) = 

f(| x |) = 

|f(x)| = 

so check at 0 & 1 using fundamental theorem of addition.

Sol.11 $f(x) = \begin{cases} 0 & ;\ x = -1 \\ \cos^{-1}\left(\text{sgn}\left(\dfrac{-4}{3x+2}\right)\right) = 0 & ;\ x < -1 \\ \cos^{-1}\left(\text{sgn}\left(\dfrac{-2}{3x+1}\right)\right) = 0 & ;\ x > -1 \end{cases}$

so continuous & differentiable at $x = -1$

$f(x) = \begin{cases} 0 & ;\ x = 1 \\ 0 & ;\ x > 1 \\ \pi/2 & ;\ x < 1 \end{cases}$

so neither continuous non differentiable at $x = 1$.

Sol.12 $f(x) = \begin{cases} 0 & ; 0 \le x < 1 \\ x & ; 1 \le x < 2 \\ 2(x-1) & ; 2 \le x < 3 \\ 3(x-1) & ; x = 3 \end{cases}$

check at x = 1 & x = 2.

Sol.13 $f'(0^+) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$

$$= \lim_{h\to 0} \frac{h.\frac{(e^h-2)}{h}+1}{h} = 1$$

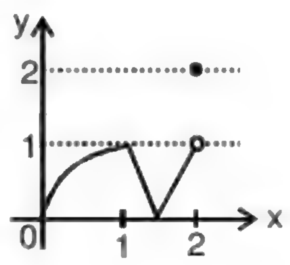
$$f'(0^-) = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h}$$

$$= \lim_{h\to 0} \frac{\frac{(-h)[e^{-1-h}-2]}{(-1-h)}-2}{-h} = \text{ND.}$$

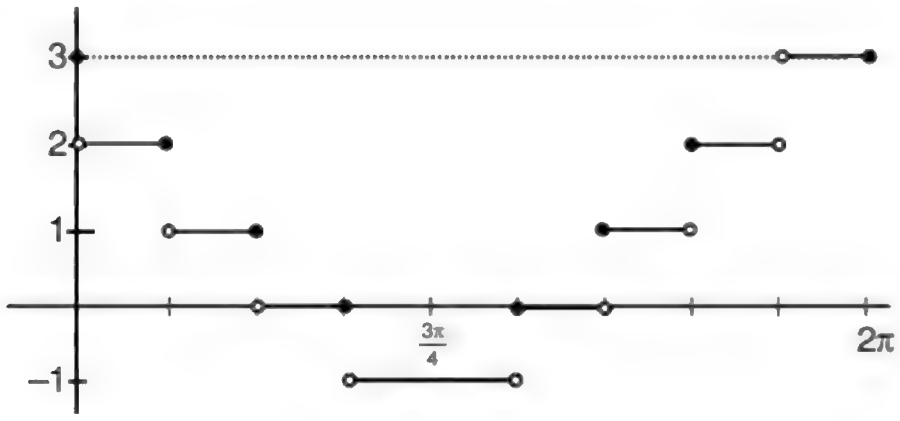
so non differentiable at x = 0

Sol.14

$$f(x) = \begin{cases} \sin\frac{\pi x}{2} & ; x < 1 \\ 3-2x & ; [1, 3/2) \\ 2x-3 & ; [3/2, 2) \\ 2 & ; x = 2 \end{cases} =$$



Sol.15



$$f(x) = \begin{cases} 3 \text{ if } \sin x = 0 \\ 2 \text{ if } -1/4 \le \sin x < 0 \\ 1 \text{ if } -1/2 \le \sin x < -1/4 \\ 0 \text{ if } -3/4 \le \sin x < -1/2 \\ -1 \text{ if } -1 \le \sin x < -3/4 \end{cases}$$

$\Rightarrow$ sum = $12\pi = \frac{k\pi}{2} \Rightarrow k = 24$

Sol.16 $f(1^-) = f(1^+) \Rightarrow b = 0$

& also $3p + q = 0$ ....(1)

$f'(3^-) = 1 = f'(3^+) = (2 \times 3)p + q$ ....(2)

from (1) & (2) $\Rightarrow p = 1/3, q = -1$.

also $f'(1^-) \ne f'(1^+)$

$\Rightarrow a \ne 1$

Sol.17 $f'(0^+) = \lim_{h\to 0} \frac{h.\frac{a^{1/h}-a^{-1/h}}{a^{1/h}+a^{-1/h}}-0}{h}$

$$= \frac{a^{1/h}-a^{-1/h}}{a^{1/h}+a^{-1/h}} \begin{cases} 0, & a = 1 \\ -1, & a \in (0,1) \\ 1, & a > 1 \end{cases}$$

$$f'(0^-) = \begin{cases} 1, & a \in (0,1) \\ 0, & a = 1 \\ -1, & a > 1 \end{cases}$$

so continuous but not differentiable for a > 1 & a < 1.

Sol.18 at x = 0

$$f(0^+) = \lim_{h\to 0} \underset{\substack{\downarrow \\ 0}}{h} \underbrace{\sin 1/h \sin\frac{1}{h.\sin 1/h}}_{\text{finite}} = 0 = f(0^-) = f(0)$$

$$f\left(\frac{1}{r\pi^+}\right) = \lim_{x\to\frac{1}{r\pi^+}} . \underset{\substack{\downarrow \\ 1/r\pi}}{x} \;\; \underset{\substack{\downarrow \\ 0}}{\sin 1/x} \underbrace{\sin\left(\frac{1}{x.\sin 1/x}\right)}_{b/w[-1,1]} = 0$$

$$= f\left(\frac{1^-}{r\pi}\right) = f\left(\frac{1}{r\pi}\right)$$

so continious between [0, 1].

for differentiability :

$$f'(0^+) = \lim_{h\to 0} \frac{\left[\underbrace{h \sin\frac{1}{h}}_{b/w[-1,1]} . \sin\frac{1}{\underbrace{h.\sin 1/h}_{0}} - 0\right]}{h}$$

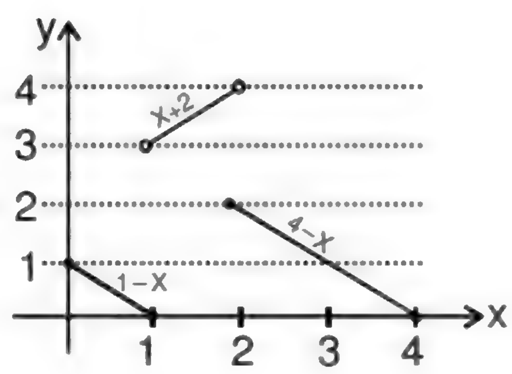
b/w [–1, 1]

= oscillating but finite

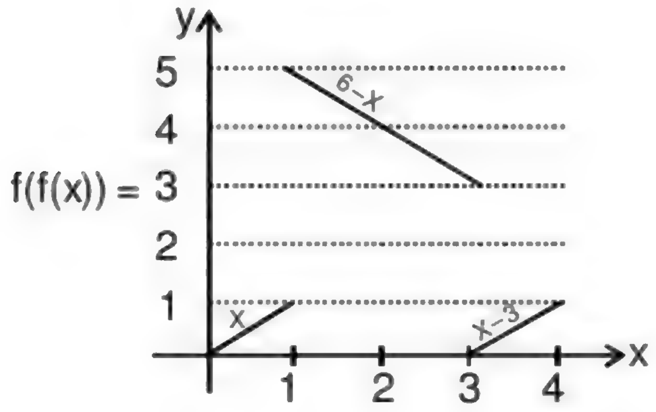
so $f'(0^+)$ = DNE

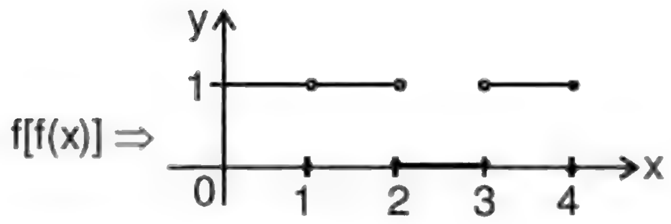
so non differentiable at x = 0.

Sol.19 $f(x) \Rightarrow$



$$f(f(x)) = \begin{cases} x & ; 0 < x < 1 \\ 6-x & ; 1 < x < 3 \\ x-3 & ; 3 < x < 4 \end{cases}$$





where [*] denotes GIF

Sol.20 $f(0) = 1\ ;\ f(0) > 0$

$$f'(x) = \lim_{h\to 0} \frac{f(x).f(h) - f(h)}{h}$$

$$= \lim_{h\to 0} f(x)\,.\underbrace{\frac{[f(h) - f(0)]}{h}}_{f'(0)} = -f(x).$$

Sol.21 $f'(0^+) = \lim_{h\to 0} \frac{f(h) - f(0)}{h}$

$$= \lim_{h\to 0} \frac{f(h) - f(kh) - f(o) + f(kh)}{h}$$

$$= \lim_{h\to 0} \frac{f(h) - f(kh)}{h} + \lim_{h\to 0} \frac{f(kh) - f(0)}{kh} \times k$$

$f'(0^+) = \alpha + k\,f'(0) \Rightarrow f'(0) = \alpha + k\,f'(0)$

$\Rightarrow\ f'(0) = \frac{\alpha}{1-k}$ $\qquad f'(0^-) = \alpha + k\,f'(0)$

Sol.22 $\lim_{x\to 0} \frac{f(x) - f(kx)}{x} = \alpha$

$$\Rightarrow \lim_{x\to 0} \frac{f'(x) - kf'(kx)}{1} = \alpha$$

$$\Rightarrow f'(0) - kf'(0) = \alpha \ \Rightarrow f'(0) = \frac{\alpha}{1-k}$$

$$f'(0^+) = \lim_{h\to 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h\to 0} \frac{f(h) - f(kh) - f(0) + f(kh)}{h}$$

$$= \lim_{h\to 0} \frac{f(h) - f(kh)}{h} + \lim_{h\to 0} \left(\frac{f(kh) - f(0)}{kh}\right) \times k$$

$$= \alpha + k\,.\,f'(0) = \alpha + \frac{k\alpha}{1-k} = \frac{\alpha}{1-k}$$

$$f'(0^-) = \lim_{h\to 0} \frac{f(-h) - f(0)}{-h}$$

$$= \lim_{h\to 0} \frac{f(-h) - f(-kh) - f(0) + f(-kh)}{-h}$$

$$= \alpha + kf'(0) = \frac{\alpha}{1-k}$$

Sol.23 $f'(0^+) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$

$$= \lim_{h\to 0} \frac{h^2\left|\cos\frac{\pi}{2h}\right| - 0}{h}$$

$$= \lim_{h\to 0} \underbrace{h}_{0} \underbrace{\left|\cos\frac{\pi}{2h}\right|}_{b/w\,[-1,1]} = 0 = f'(0^-)$$

so $f'(0) = 0$

$$f'\left(\frac{1}{3}^-\right) = -\frac{\pi}{2} \text{ and } f'\left(\frac{1}{3}^+\right) = \frac{\pi}{2}$$

$f'(x)$ fails to exist at $\frac{1}{2n+1}$ where $n \in I$.

Sol.24 $f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$

$$= \lim_{h\to 0} \frac{f[(h^{1/n})] - 0}{h} = \lim_{h\to 0} \frac{f((h^{1/n}))^n}{(h^{1/n})^n} = (f'(0))^n$$

$\Rightarrow f'(0)\,[f'(0)^{n-1} - 1] = 0 \ \Rightarrow f'(0) = 0$ or ± 1
but $f'(0) \geq 0 \Rightarrow f'(0) = 0$ or 1.

Now $f'(x) = \lim_{h\to 0} \frac{f(x + (h^{1/n})^n) - f(x)}{h}$

$$= \lim_{h\to 0} \frac{f(h^{1/n})^n}{(h^{1/n})^n} = (f'(0))^n$$

If $f'(0) = 0$ then $f'(x) = 0$ so $f'(0) \neq 0$
If $f'(0) = 1 \Rightarrow f(x) = x + c$
(using boundry condn $c = 0$)
$\Rightarrow$ $f(x) = x$ so $f(10) = 10$
$f(x) - f(y) \geq \ell n\, x - \ell n\, y + x - y$
put $x = x + h,\ y = x$

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\ell n(x+h) - \ell n x + h}{h}$$

$$\Rightarrow f'(x) \geq -\frac{1}{x} + 1$$

If $x = x - h$ & $y = x$

$$\lim_{h\to 0} \frac{f(x-h) - f(x)}{-h} \geq \lim_{h\to 0} \frac{f(x-h) - \ell n\, x - h}{-h}$$

$$\Rightarrow f'(x) \geq \frac{1}{x} + 1 = g(x) \Rightarrow \sum_{n=1}^{\infty} g\left(\frac{1}{n}\right) = \sum_{n=1}^{\infty} (n+1) = 5150$$

EXERCISE – V HINTS & SOLUTIONS

Sol.1 $f(x) = [x]^2 - [x^2]$

$\left.\begin{matrix} RHL = 0 \\ LHL = 1 \end{matrix}\right\}$ at x = 0 $\left.\begin{matrix} RHL = 0 \\ LHL = 0 \end{matrix}\right\}$ at x = 1

Sol.2 RHL = $\lim_{x\to 0} \frac{(x+c)^{1/3}-1}{(x+1)^{1/2}-1}$; $\frac{c^{1/3}-1}{0}$ form

$\Rightarrow c = 1$

$= \lim_{x\to 0} \frac{(x+c)^{1/3}-1}{(x+1)^{1/2}-1}$

$= \lim_{x\to 0} \frac{1+\frac{1}{3}x-1}{1+\frac{1}{2}x-1} = \frac{2}{3} \Rightarrow \quad b = \frac{2}{3}$

LHL = $\lim_{x\to 0} (1+ax)^{1/x} = e^{\lim_{x\to 0} a} = e^a$

$e^a = \frac{2}{3} \Rightarrow a = \ell n \frac{2}{3}$

$a = \ell n \frac{2}{3}$; $b = \frac{2}{3}$; $c = 1$

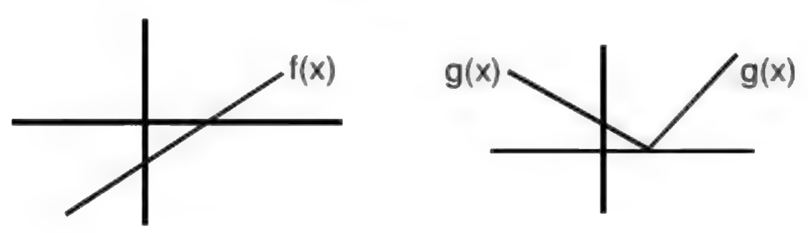
Sol.3 RHL $\lim_{x\to 1^+} \frac{e^{1/x-1}-2}{e^{1/x-1}+2} = \frac{1-0}{1+0} = 1$

LHL $\lim_{x\to 1^-} \frac{e^{1/x-1}-2}{e^{1/x-1}+2} = \frac{0-2}{0+2} = -1$

discont at x = 1

Sol.4 $f : R \to R$ $\qquad g : R \to R$

$g(x) = |f(x)|$



Sol.5 $f(x) = \frac{x}{1+|x|}$; $x \ge 1$ or $x \le -1$

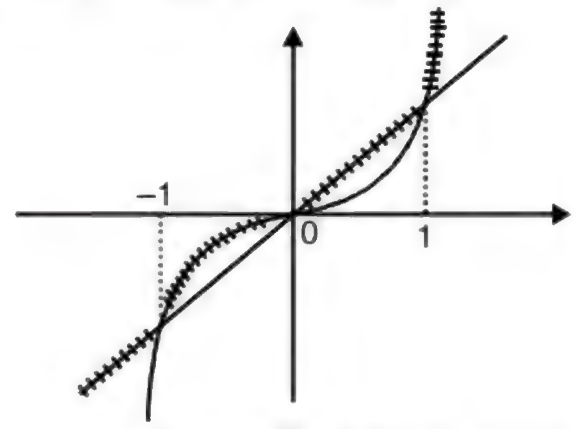
$= \frac{x}{1-|x|}$, $-1 < x < 1$

$f(x) = \frac{x}{1+x}$; $x \ge 1$, $-1 < x < 0$

$= \frac{x}{1-x}$; $x \le -1$, $0 \le x < 1$

Discont. at x = 1 and x = –1 hence not differentiable at x = 1, –1 and cont. & derivaiable at x = 0

Sol.6 **(a)** $f : R \to R$ $\qquad f(x) = \max[x, x^3]$



Non- diff at x = 0, 1, –1

(b) $f(x) = [x] \sin \pi x$

If x is just less than k, [x] = k – 1

$f(x) = (k-1) \sin \pi x$,

LHD of f(x) = $\lim_{x\to k} \frac{(k-1)\sin \pi x - k \sin \pi k}{x-k}$

$= \lim_{h\to 0} \frac{(k-1)\sin \pi(k-h)}{-h}$ $\qquad (x = k - h)$

$= \lim_{h\to 0} \frac{(k-1)(-1)^{k-1}\sin h\pi}{-h} = (-1)^k (k-1)\pi$

(c) RHD of $\sin(|x|) - |x| = \lim_{h\to 0} \frac{\sin h - h}{h} = 1 - 1 = 0$

$(\because f(0) = 0)$

LHD of $\sin(|x|) - |x|$

$= \lim_{h\to 0} \frac{\sin|-h| - |-h|}{-h} = \frac{\sin h - h}{-h} = 0$

Sol.7 If $\lim_{x\to \alpha^+} g(x) = \lim_{x\to \alpha^-} g(x) = g(\alpha)$ &

$f(x) - f(\alpha) = g(x)(x-\alpha)$ ....(1)

so $f(x) - f(\alpha) = g(x)(x-\alpha)$ & g(x) must be continous as $f'(\alpha^+) = f'(\alpha^-)$

Sol.8 $f(x) = \tan^{-1} x$ $\qquad -1 \le x \le 1$

$= \frac{1}{2}(x-1)$ $\qquad x > 1$

$= \frac{1}{2}(-x-1)$ $\qquad x < -1$

f(x) is discontinuous at x = 1, –1 hence non-diff. at x = 1, –1

Sol.9 $f : R \to R$ $\quad f(1) = 3 \,;\, f'(1) = 6$

$$\lim_{x\to 0}\left[\frac{f(1+x)}{f(1)}\right]^{1/x} = e^{\lim\limits_{x\to 0}\frac{1}{x}\left(\frac{f(1+x)-f(1)}{f(1)}\right)}$$

$$= e^{\frac{f'(1)}{f(1)}} = e^2$$

Sol.10 $f(x) = \begin{cases} x+a \ , & x<0 \\ x-1 \ , & x\ge 1 \\ 1-x \ , & 0\le x<1 \end{cases}$; $g(x) = \begin{cases} x+1 \ , & x<0 \\ (x-1)^2+b \ , & x\ge 0 \end{cases}$

$$gof(x) = g[f(x)] = \begin{cases} f(x)+1 \ , & f(x)<0 \\ [f(x)-1]^2+b \ , & f(x)\ge 0 \end{cases}$$

Now, $f(x) < 0$

$$\Rightarrow \begin{cases} x+a<0 \ , & x<0 \\ x-1<0 \ , & x\ge 1 \\ 1-x<0 \ , & 0\le x<1 \end{cases}$$

$\Rightarrow x < -a$ when $x < 0$

$x < 1$ when $x \ge 1$

$x > 1$ when $0 \le x < 1$

The last two cases are not possible

so, $f(x) < 0$ if $x < -a$

a is positive

$f(x) < 0$ for $x < -a$

$\Rightarrow f(x) \ge 0$ for $x > -a$

Now,

$$gof(x) = \begin{cases} f(x)+1 \ , & x<-a \\ [f(x)-1]^2+b \ , & x\ge -a \end{cases} \quad \text{where } f(x) = x + a$$

$$gof(x) = \begin{cases} x+a+1 \ , & x<-a \\ (x+a-1)^2+b \ , & -a\le x<0 \end{cases}$$

$= (1 - x - 1)^2 + b, \ 0 \le x < 1$

$= x^2 + b, \quad 0 \le x < 1$

$gof(x) = (x - 1 - 1)^2 + b, \ x \ge 1$

$= (x - 2)^2 + b \ , \ x \ge 1$

since, gof is continuous for all real x, therefore, $(a - 1)^2 + b = b \Rightarrow a = 1$, b is any real number.

for $a = 1$, $b \in R$, gof is continuous

$$gof(x) = \begin{cases} x+2 \ , & x<-a \\ x^2+b \ , & -a\le x<1 \\ (x-2)^2+b \ , & x\ge 1 \end{cases}$$

so gof is differentiable at $x = 0$ if $a = 1$, $b \in R$

Sol.11 $f'(a^-) = \lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h} = 0, \ x \in (0, 2a)$

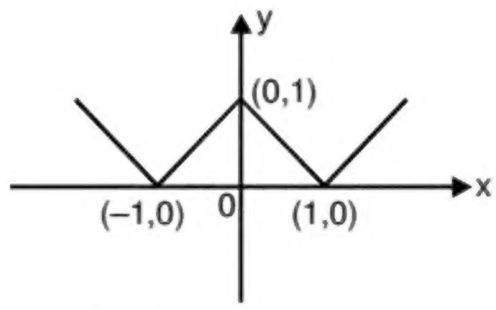
Now $f'(-a^-) = \lim\limits_{h\to 0}\dfrac{f(-a+h)-f(-a)}{h}$

$= \lim\limits_{h\to 0}\dfrac{-f(a-h)+f(a)}{h}$, f is an odd function

$= \lim\limits_{h\to 0}\dfrac{-f(a+h)+f(a)}{h}$; $f(x) = f(2a - x), x \in (a, 2a)$

$= -\lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h} = 0$

Sol.12 **(a)** $y = ||x| - 1|$



Non-differentiable at $x = 1, 0, -1$

(b) $|f(x_1) - f(x_2)| < (x_1 - x_2)^2$

$$\Rightarrow \lim_{x_1\to x_2}\left|\frac{f(x_1)-f(x_2)}{x_1-x_2}\right| < \lim_{x_1\to x_2}|x_1 - x_2|$$

$\Rightarrow |f'(x)| < \delta \quad \Rightarrow \quad f'(x) = 0$

Hence f(x) is a constant function and P(1, 2) lies on the course.

$\Rightarrow f(x) = 2$ is the curve.

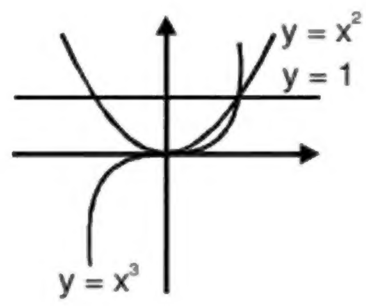
Hence the equation of tangent is $y - 2 = 0$

Sol.13 $f(x) = \min.\{1, x^2, x^3\}$

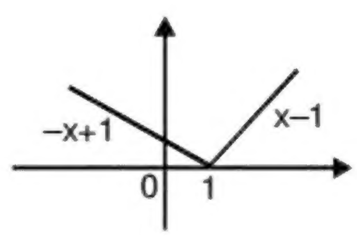
$f(x) = \begin{cases} x^3 \ , & x\le 1 \\ 1 \ , & x>1 \end{cases}$

f(x) is continuous $\forall\, x\in R$

and non-diff. at $x = 1$



Sol.14 From graph, $p = -1$



$\Rightarrow \lim\limits_{x\to 1^+} g(x) = -1$

$\Rightarrow \lim\limits_{h\to 0} g(1 + h) = -1$

$$\Rightarrow \lim_{h\to 0}\left(\frac{h^n}{\log\cos^m h}\right) = -1$$

$$\Rightarrow \lim_{h\to 0} \frac{n.h^{n-1}}{m.(-\tan h)} = \left(-\frac{n}{m}\right) \lim_{h\to 0}\left(\frac{h^{n-1}}{\tan h}\right) = -1$$

which holds if n = m = 2

Sol.15 $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$$= \lim_{h\to 0} \frac{f(x)+f(h)-f(x)}{h} = f'(0)$$

$\{\because f(0)=0\}$ = k(let)

$\Rightarrow$ f(x) = kx + c ; c = 0

hence f(x) = kx

Sol.16 Let $y = \frac{b-x}{1-bx} \Rightarrow x = \frac{b-y}{1-by}$

so $f^{-1}(x) = \frac{b-x}{1-bx}$

so $f = f^{-1}$ on (0, 1)

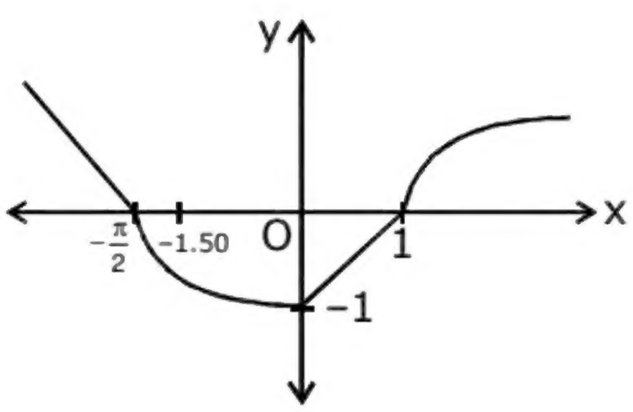
$f'(0) = 1 - b^2$ & $f'(b) = \frac{1}{1-b^2}$

so $f'(b) = \frac{1}{f'(0)}$

also f^{-1} is differentiable on (0, 1)

Sol.17

graph of f(x) :



Answer Ex–I SINGLE CORRECT (OBJECTIVE QUESTIONS)

1. A	**2.** B	**3.** D	**4.** C	**5.** C	**6.** B	**7.** B
8. B	**9.** B	**10.** B	**11.** B	**12.** D	**13.** B	**14.** D
15. D	**16.** D	**17.** C	**18.** B	**19.** D	**20.** C	**21.** D
22. B	**23.** B	**24.** C	**25.** C	**26.** C	**27.** C	**28.** C
29. C	**30.** D	**31.** A	**32.** D	**33.** C	**34.** D	**35.** D
36. B	**37.** B	**38.** C	**39.** C	**40.** B	**41.** D	**42.** D
43. D	**44.** B	**45.** C	**46.** D	**47.** A	**48.** B	**49.** C
50. B	**51.** B	**52.** C	**53.** A	**54.** B	**55.** A	**56.** D
57. A	**58.** D	**59.** C	**60.** B	**61.** A	**62.** D	**63.** D
64. D	**65.** D					

Answer Ex–II MULTIPLE CORRECT (OBJECTIVE QUESTIONS)

1. ABC	**2.** BCD	**3.** ABC	**4.** ABD	**5.** ABC	**6.** BD	**7.** ABC
8. AC	**9.** ABD	**10.** BD	**11.** ABD	**12.** ABD	**13.** AC	**14.** B, C
15. C, D	**16.** AC	**17.** AB	**18.** BD	**19.** BD	**20.** ABC	**21.** BD
22. ABD	**23.** ABD	**24.** ABD				

Answer Ex–III SUBJECTIVE QUESTIONS (CONTINUITY)

1. –1 **2.** a = 0, b = 1 **3.** $f(0^+) = -2$; $f(0^-) = 2$ hence f(0) not possible to define

4. (a) –2, 2, 3 **(b)** K = 5 **(c)** even **5.** $y_n(x)$ is continuous at x = 0 for all n and y(x) is dicontinuous at x = 0

6. f is cont. in $-1 \le x \le 1$ **7.** P not possible. **8. (a)** $4 - 3\sqrt{2} + a$, **(b)** a = 3

9. g(x) = 2 + x for $0 \le x \le 1$, 2 – x for $1 < x \le 2$, 4 – x for $2 < x \le 3$, g is discontinuous at x = 1 & x = 2

10. A = 1 ; f(2) = 1/2 **11.** a = 0 ; b = –1 **12.** a = 1/2, b = 4 **13.** a = – 3/2, b ≠ 0, c = 1/2

14. A = –4, B = 5, f(0) = 1 **15.** $f(0^+) = \frac{\pi}{2}$; $f(0^-) = \frac{\pi}{4\sqrt{2}}$ $\Rightarrow$ f is discont. at x = 0 ;

$g(0^+) = g(0^-) = g(0) = \pi/2 \Rightarrow$ g is cont. at x = 0

16. the function f is continuous everywhere in [0, 2] except for x = 0, $\frac{1}{2}$, 1 & 2

17. discontinuous at all integral values in [–2, 2]

18. locus $(a, b) \to x, y$ is y = x – 3 excluding the points where y = 3 intersects it.

19. c = 1, a, b ∈ R **20.** 5 **21.** $\frac{1}{60}$

24. k = 0 ; $g(x) = \begin{cases} \ell n(\tan x) & \text{if } 0 < x < \frac{\pi}{4} \\ 0 & \text{if } \frac{\pi}{4} \le x < \frac{\pi}{2} \end{cases}$. Hence g(x) is continuous everywhere.

25. g(x) = 4(x + 1) and limit $= -\frac{39}{4}$

27. $a = \frac{1}{\sqrt{2}}$, $g(0) = \frac{(\ell n\, 2)^2}{8}$

Answer Ex–IV SUBJECTIVE QUESTIONS (DIFFERENTIABILITY)

1. f(x) is conti. but not derivable at x = 0

2. conti. $\forall$ $x \in R$, not diff. at x = 0, 1 & 2

3. $0 < n \le 1$

4. conti. but not diff. at x = 0; diff. & conti. at $x = \pi/2$

5. conti. but not diff. at x = 0

7. f is cont. but not diff. at x = 0

8. $f'(1^+) = 3$, $f'(1^-) = -1$

9. a = 1/2, b = 3/2

10. not derivable at x = 0 & x = 1

11. f is cont. & derivable at x = –1 but f is neither cont. nor derivable at x = 1

12. discontinuous & not derivable at x = 1, continuous but not derivable at x = 2

13. not derivable at x = 0

14. f is conti. at x = 1, 3/2 & disconti. at x = 2, f is not diff. at x = 1, 3/2, 2

15. 24

16. $a \ne 1$, b = 0, $p = \frac{1}{3}$ and q = – 1

17. If $a \in (0, 1)$ $f'(0^+) = -1$; $f'(0^-) = 1$ $\Rightarrow$ continuous but not derivable

If a = 1 ; f(x) = 0 which is constant $\Rightarrow$ continuous but not derivable

If a > 1 $f'(0^-) = -1$; $f'(0^+) = 1$ $\Rightarrow$ continuous but not derivable

18. conti. in $0 \le x \le 1$ & not diff. at x = 0

19. f is conti. but not diff. at x = 1, disconti. at x = 2 & x = 3. cont. & diff. at all other points

20. $f'(x) = -f(x)$

21. continuous but not derivable at $x = \sqrt{2}$

22. $f'(0) = \frac{\alpha}{1-k}$

23. **(a)** $f'(0) = 0$, **(b)** $f'\left(\frac{1^-}{3}\right) = -\frac{\pi}{2}$ and $f'\left(\frac{1^+}{3}\right) = \frac{\pi}{2}$, **(c)** $x = \frac{1}{2n+1}$ $n \in I$

24. f(x) = x $\Rightarrow$ f(10) = 10

25. 5150

Answer Ex–V JEE PROBLEMS

1. D

2. $a = \ell n\frac{2}{3}$; $b = \frac{2}{3}$; c = 1

3. Discontinuous at x = 1; $f(1^+) = 1$ and $f(1^-) = -1$

4. C

5. Discont. Hence hot deri. at x = 1 & –1. Cont. & deri. at x = 0

6. **(a)** D, **(b)** A, **(c)** D

8. D

9. C

10. a = 1 ; b = 0 $(gof)'(0) = 0$

11. $f'(a^-) = 0$

12. **(a)** A, **(b)** y – 2 = 0

13. A, C

14. C

15. B,C

16. C,D

17. A,B,C,D